Four-Dimensional Input-Output Calculator

Mathematical Description, Calculation Steps and Operation

English

Wolfgang Boehme

wolfgang.boehme@economic-modeling.de

www.economic-modeling.de

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1 Introduction and Overview

This document describes the mathematical foundations and the operation of the web-based four-dimensional input-output calculator and works through Example 1 step by step. The tool is divided into five tabs:

- **Tab 1** — General four-dimensional model (basis for tabs 2 and 3)
- **Tab 2** — Hierarchical structure: derivation of the models M1–M9
- **Tab 3** — Partial input-output tables (split into two groups of goods)
- **Tab 4** — Make-use matrices (independent data basis)
- **Tab 5** — Derivation of the material-value balance (own data input, optionally from tab 1)

Operating note

Tab 1 is processed first (load or enter data, *Calculate*). Tabs 2 and 3 use the intermediate matrix X_{ijkl} and the gross output vector X_{ik} calculated there. Tab 4 works with the use and make matrices, Tab 5 with its own table (which can be taken over from tab 1).

1.1 Example datasets

The examples are stored in the central databox (tag "Four-dimensional") and are loaded in each tab via *Load data from databox*.

Name in the tool	Content	Tabs
Example 1 (5 goods, 2 sectors)	numerical example; $N_1 = \{1, 2, 3\}$	1, 2, 3, 5
Example 2 (5 goods, 2 sectors)	own, consistent example; $N_1 = \{1, 2, 3\}$	1, 2, 3, 5
Example 3 (3 goods, 2 sectors)	reduced example with 3 goods	1, 2, 3, 5
Example 4 (3 goods, 2 sectors)	second own example	1, 2, 3, 5
Make-Use Example 1 (3 goods, 3 sectors)	numerical example (large values)	4
Make-Use Example 2 (3 goods, 3 sectors)	own, consistent example	4

Example 1 (tabs 1, 2, 3, 5) serves as the running numerical example, **Make-Use Example 2** for Tab 4.

1.2 Notation and ordering

Symbol	Meaning
i, j	goods indices, $i, j = 1, \ldots, n$
k, l	sector indices, $k, l = 1, \ldots, m$
(i, k)	combined index: good i produced in sector k (Gi-Bk in the tool)
X_{ijkl}	delivery of good i from sector k for the production of good j in sector l
X_{ik}	gross output of good i in sector k
Y_{ik}	final demand for good i from sector k

The combined indices are ordered **by sector**: G1-B1, G2-B1, …, Gn-B1, G1-B2, … (B stands for sector). A CSV file has the form

```
n=5;m=2
;G1-B1;G2-B1;...;G5-B2
G1-B1;20;20;...
...
Y;20;20;...
```

If a file is ordered by goods (G1-B1, G1-B2, G2-B1, …), the tool re-sorts it on the basis of the labels and points this out.

2 Tab 1 — General Four-Dimensional Model

2.1 Basic equations

Balance equation (model M1) for each good i from sector k:

$$X_{ik} = \sum_{j=1}^{n} \sum_{l=1}^{m} X_{ijkl} + Y_{ik}, \qquad i = 1, \ldots, n,\ k = 1, \ldots, m. \tag{1}$$

Input coefficient:

$$a_{ijkl} = \frac{X_{ijkl}}{X_{jl}}. \tag{2}$$

Linear system of equations and Leontief inverse:

$$\mathbf{X} = \mathbf{A}\mathbf{X} + \mathbf{Y} \quad \Longleftrightarrow \quad \mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{Y} = \mathbf{L}\mathbf{Y}, \tag{3}$$

with the $(nm) \times (nm)$ matrix $\mathbf{A}$. Weight matrix (share of the supplying sector k in all deliveries of good i to the use (j, l)):

$$w_{ijkl} = \frac{X_{ijkl}}{\sum_{k'} X_{ijk'l}}, \qquad \sum_{k} w_{ijkl} = 1. \tag{4}$$

Valuation with one price p_{ik} per good and sector ($n \cdot m$ prices for n goods and m sectors, e. g. 6 for 3 goods and 2 sectors, 10 for 5 goods and 2 sectors); the same good may have a different price in different sectors:

$$X^{p}_{ik} = p_{ik}\, X_{ik}, \qquad X^{p}_{ijkl} = p_{ik}\, X_{ijkl}, \qquad a^{p}_{ijkl} = a_{ijkl}\, \frac{p_{ik}}{p_{jl}}, \qquad \mathbf{A}^{p} = \langle p \rangle \mathbf{A} \langle p \rangle^{-1}. \tag{5}$$

2.2 Calculation steps for Example 1

Step 1 — Data. Intermediate matrix $Z = (X_{ijkl})$, final demand Y and gross output X (calculated in step 2):

	G1-B1	G2-B1	G3-B1	G4-B1	G5-B1	G1-B2	G2-B2	G3-B2	G4-B2	G5-B2	Y	X
G1-B1	20	20	0	1	4	30	20	0	0	0	20	115
G2-B1	0	14	0	0	0	0	6	10	2	0.5	20	52.5
G3-B1	0	0	20	0	0	0	0	20	5	0	50	95
G4-B1	4	0	0	0	2	0	0	10	10	10	5	41
G5-B1	3	0	0	3	1	0	5	0	5	5	5	27
G1-B2	0	20	0	0	0	100	40	0	0	0	25	185
G2-B2	0	0	0	0	0	0	10	10	0	0	27.5	47.5
G3-B2	0	0	25	25	0	0	0	25	3	2	45	125
G4-B2	10	0	4	4	0	8	2	5	0	0	20	53
G5-B2	0	5	3	3	5	1	4	5	0	0	5	31

Step 2 — Gross output according to Eq. (1) as row sum plus final demand, e. g.

$$\begin{aligned} X_{1,1} &= 20 + 20 + 0 + 1 + 4 + 30 + 20 + 0 + 0 + 0 + 20 = 115, \\ X_{2,1} &= 0 + 14 + 0 + 0 + 0 + 0 + 6 + 10 + 2 + 0.5 + 20 = 52.5, \\ X_{3,2} &= 0 + 0 + 25 + 25 + 0 + 0 + 0 + 25 + 3 + 2 + 45 = 125. \end{aligned}$$

In total $X = (115;\ 52.5;\ 95;\ 41;\ 27;\ 185;\ 47.5;\ 125;\ 53;\ 31)^{\top}$.

Step 3 — Coefficient matrix according to Eq. (2): divide each column of Z by the output of the receiving row, e. g.

$$a_{1,2,1,1} = \frac{X_{1,2,1,1}}{X_{2,1}} = \frac{20}{52.5} = 0.381, \qquad a_{2,2,1,2} = \frac{6}{47.5} = 0.126, \qquad a_{3,4,2,1} = \frac{25}{41} = 0.610.$$

	G1-B1	G2-B1	G3-B1	G4-B1	G5-B1	G1-B2	G2-B2	G3-B2	G4-B2	G5-B2
G1-B1	0.174	0.381	0	0.024	0.148	0.162	0.421	0	0	0
G2-B1	0	0.267	0	0	0	0	0.126	0.08	0.038	0.016
G3-B1	0	0	0.211	0	0	0	0	0.16	0.094	0
G4-B1	0.035	0	0	0	0.074	0	0	0.08	0.189	0.323
G5-B1	0.026	0	0	0.073	0.037	0	0.105	0	0.094	0.161
G1-B2	0	0.381	0	0	0	0.541	0.842	0	0	0
G2-B2	0	0	0	0	0	0	0.211	0.08	0	0
G3-B2	0	0	0.263	0.61	0	0	0	0.2	0.057	0.065
G4-B2	0.087	0	0.042	0.098	0	0.043	0.042	0.04	0	0
G5-B2	0	0.095	0.032	0.073	0.185	0.005	0.084	0.04	0	0

Column sums above 1 (G2-B1: 1.124, G2-B2: 1.832) are admissible here: the table is set up in physical units, quantities of different goods cannot be added. Decisive for solvability is the spectral radius $\rho(\mathbf{A}) = 0.665 < 1$; hence $\mathbf{L}$ exists and is non-negative.

Step 4 — Leontief inverse $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$:

	G1-B1	G2-B1	G3-B1	G4-B1	G5-B1	G1-B2	G2-B2	G3-B2	G4-B2	G5-B2
G1-B1	1.242	0.9	0.108	0.245	0.239	0.452	1.343	0.284	0.129	0.15
G2-B1	0.018	1.393	0.077	0.144	0.031	0.017	0.269	0.206	0.102	0.087
G3-B1	0.032	0.046	1.391	0.232	0.043	0.031	0.085	0.33	0.2	0.104
G4-B1	0.085	0.132	0.102	1.203	0.189	0.061	0.217	0.211	0.271	0.434
G5-B1	0.057	0.079	0.043	0.162	1.105	0.037	0.263	0.078	0.146	0.237
G1-B2	0.031	1.181	0.167	0.31	0.059	2.203	2.588	0.457	0.15	0.158
G2-B2	0.009	0.014	0.056	0.104	0.018	0.007	1.29	0.156	0.036	0.047
G3-B2	0.086	0.141	0.551	1.023	0.181	0.069	0.234	1.541	0.355	0.461
G4-B2	0.123	0.15	0.11	0.207	0.052	0.145	0.317	0.147	1.068	0.087
G5-B2	0.024	0.172	0.094	0.19	0.232	0.029	0.225	0.137	0.081	1.11

Back-calculation check: $\mathbf{L}\,Y = X$ reproduces exactly

$$X = (115;\ 52.5;\ 95;\ 41;\ 27;\ 185;\ 47.5;\ 125;\ 53;\ 31)^{\top}.$$

Step 5 — Weight matrix according to Eq. (4). Example column G2-B1, good 1: good 1 is delivered to (G2,B1) from sector 1 with 20 and from sector 2 with 20, so $w_{1,2,1,1} = w_{1,2,2,1} = 20/40 = 0.5$. Column G1-B1, good 4: 10 from sector 2 and 4 from sector 1, so $w_{4,1,1,1} = 4/14 = 0.286$, $w_{4,1,2,1} = 10/14 = 0.714$. If no sector delivers, $w = 0$.

	G1-B1	G2-B1	G3-B1	G4-B1	G5-B1	G1-B2	G2-B2	G3-B2	G4-B2	G5-B2
G1-B1	1	0.5	0	1	1	0.231	0.333	0	0	0
G2-B1	0	1	0	0	0	0	0.375	0.5	1	1
G3-B1	0	0	0.444	0	0	0	0	0.444	0.625	0
G4-B1	0.286	0	0	0	1	0	0	0.667	1	1
G5-B1	1	0	0	0.5	0.167	0	0.556	0	1	1
G1-B2	0	0.5	0	0	0	0.769	0.667	0	0	0
G2-B2	0	0	0	0	0	0	0.625	0.5	0	0
G3-B2	0	0	0.556	1	0	0	0	0.556	0.375	1
G4-B2	0.714	0	1	1	0	1	1	0.333	0	0
G5-B2	0	1	1	0.5	0.833	1	0.444	1	0	0

Valuation. The default is $p_{ik} = 1$ for all goods and sectors; then the value and the quantity representation are identical. With other prices Eq. (5) applies, e. g. with $p_{1,1} = 2$ (good 1 from sector 1) and $p_{ik} = 1$ otherwise: $X^{p}_{1,1} = 2 \cdot 115 = 230$, row G1-B1 of A^{p} doubled, column G1-B1 halved (the diagonal element $a_{1,1,1,1}$ remains unchanged).

Operating note

Operating Tab 1: (1) *Load data from databox* → select an example; the table is filled and calculated immediately. Alternatively define the structure, *Create tables* and enter data or load a CSV, then (2) *Calculate* — yields X, A, L, W. When new data are loaded, all earlier results are discarded. (3) Optional: *Calculate forecast* with changed final demand; enter prices per good and sector (fields G1-B1, G2-B1, ...) and *Apply prices*. (4) Export individually or as an overall CSV. (5) *Save current result in databox* stores n, m, Z, Y as

well as A and Y in the databox. Static datasets can be loaded under "More datasets" as n goods with one sector.

3 Tab 2 — Hierarchical Structure (Models M1–M9)

3.1 Classification principle

Eight further models can be derived from M1, depending on whether both characteristics (good *and* sector), only the good or only the sector are retained on the input and the output side:

	Output: j,l	Output: j	Output: l
Input: i,k	M1 $i,k \to j,l$	M2 $i,k \to j$	M3 $i,k \to l$
Input: i	M4 $i \to j,l$	M5 $i \to j$	M6 $i \to l$
Input: k	M7 $k \to j,l$	M8 $k \to j$	M9 $k \to l$

Only M1, M5 and M9 are square, uniquely solvable Leontief systems; the other six are rectangular.

3.2 Aggregation rules

Numerator and denominator of Eq. (2) are summed over the omitted indices. M4 (input only good):

$$a_{i,jl} = \frac{\sum_k X_{ijkl}}{X_{jl}} = \sum_k a_{ijkl}. \tag{6}$$

M5 (input and output only good), with $X_{j\cdot} = \sum_l X_{jl}$:

$$a_{ij} = \frac{\sum_k \sum_l X_{ijkl}}{X_{j\cdot}} = \sum_l a_{i,jl}\, d_{jl}, \qquad d_{jl} = \frac{X_{jl}}{X_{j\cdot}}. \tag{7}$$

M9 (input and output only sector), with $X_{\cdot l} = \sum_j X_{jl}$:

$$a_{kl} = \frac{\sum_i \sum_j X_{ijkl}}{X_{\cdot l}}. \tag{8}$$

3.3 Calculation steps for Example 1

M4 — sum the rows over the sectors k (e. g. good 1 to G2-B1: $20 + 20 = 40$), then divide by X_{jl} ($40/52.5 = 0.762$). Intermediate inputs:

	G1-B1	G2-B1	G3-B1	G4-B1	G5-B1	G1-B2	G2-B2	G3-B2	G4-B2	G5-B2
G1	20	40	0	1	4	130	60	0	0	0
G2	0	14	0	0	0	0	16	20	2	0.5
G3	0	0	45	25	0	0	0	45	8	2
G4	14	0	4	4	2	8	2	15	10	10
G5	3	5	3	6	6	1	9	5	5	5

Coefficients:

	G1-B1	G2-B1	G3-B1	G4-B1	G5-B1	G1-B2	G2-B2	G3-B2	G4-B2	G5-B2
G1	0.174	0.762	0	0.024	0.148	0.703	1.263	0	0	0
G2	0	0.267	0	0	0	0	0.337	0.16	0.038	0.016
G3	0	0	0.474	0.61	0	0	0	0.36	0.151	0.065
G4	0.122	0	0.042	0.098	0.074	0.043	0.042	0.12	0.189	0.323
G5	0.026	0.095	0.032	0.146	0.222	0.005	0.189	0.04	0.094	0.161

M5 — additionally sum the columns over l. Output per good (e. g. $X_{1\cdot} = 115 + 185 = 300$) and final demand per good:

$$X_{M5} = (300;\ 100;\ 220;\ 94;\ 58)^\top, \qquad Y_{M5} = (45;\ 47.5;\ 95;\ 25;\ 10)^\top,$$

$$Z_{M5} = \begin{pmatrix} 150 & 100 & 0 & 1 & 4 \\ 0 & 30 & 20 & 2 & 0.5 \\ 0 & 0 & 90 & 33 & 2 \\ 22 & 2 & 19 & 14 & 12 \\ 4 & 14 & 8 & 11 & 11 \end{pmatrix},$$

$$A_{M5} = \begin{pmatrix} 0.5 & 1 & 0 & 0.0106 & 0.069 \\ 0 & 0.3 & 0.0909 & 0.0213 & 0.0086 \\ 0 & 0 & 0.4091 & 0.3511 & 0.0345 \\ 0.0733 & 0.02 & 0.0864 & 0.1489 & 0.2069 \\ 0.0133 & 0.14 & 0.0364 & 0.117 & 0.1897 \end{pmatrix},$$

$$L_{M5} = \begin{pmatrix} 2.0627 & 3.0221 & 0.5387 & 0.3682 & 0.3246 \\ 0.024 & 1.4806 & 0.254 & 0.1514 & 0.0672 \\ 0.1287 & 0.2681 & 1.8724 & 0.8224 & 0.3035 \\ 0.2094 & 0.4141 & 0.2857 & 1.3582 & 0.3812 \\ 0.0741 & 0.3774 & 0.178 & 0.2653 & 1.3197 \end{pmatrix}.$$

Check: $L_{M5}\, Y_{M5} = (300;\ 100;\ 220;\ 94;\ 58)^\top$.

M9 — sum rows and columns per sector. Sector output $X_{.l} = (330.5;\ 441.5)^\top$, final demand per sector $(100;\ 122.5)^\top$:

$$Z_{M9} = \begin{pmatrix} 92 & 138.5 \\ 104 & 215 \end{pmatrix}, \quad A_{M9} = \begin{pmatrix} 0.2784 & 0.3137 \\ 0.3147 & 0.487 \end{pmatrix}, \quad L_{M9} = \begin{pmatrix} 1.8896 & 1.1554 \\ 1.159 & 2.6579 \end{pmatrix}.$$

Check: $L_{M9}\, Y_{M9} = (330.5;\ 441.5)^\top$. For physical units, sums over different goods can only be interpreted arithmetically, not economically (meaningful in value terms with prices).

Operating note

Operating Tab 2: After *Calculate* in Tab 1: *Derive all models M2–M9.* For each model the intermediate scheme and the coefficient matrix are shown, for M5/M9 also the Leontief inverse. Export individually or as a whole.

4 Tab 3 — Partial Input-Output Tables

4.1 Split into two groups of goods

The goods are partitioned into $N_1 \cup N_2 = \{1, \ldots, n\}$, $N_1 \cap N_2 = \emptyset$. Deliveries to the other group are assigned to final demand:

$$Y^*_{ik} = Y_{ik} + \sum_l \sum_{j \in N_2} X_{ijkl}, \qquad i \in N_1, \tag{9}$$

purchases from the other group are shown as primary input:

$$P^*_{jl} = \sum_k \sum_{i \in N_2} X_{ijkl}, \qquad j \in N_1. \tag{10}$$

The partial table for N_1 is an independent Leontief system:

$$X_{ik} = \sum_{j \in N_1} \sum_l a_{ijkl}\, X_{jl} + Y^*_{ik}, \qquad i \in N_1. \tag{11}$$

The same applies to N_2 with the roles reversed.

4.2 Calculation steps for Example 1 with $N_1 = \{1, 2, 3\}$, $N_2 = \{4, 5\}$

Step 1 — modified final demand. Row G1-B1: deliveries to N_2 are $X_{1,4,1,1} = 1$, $X_{1,5,1,1} = 4$, $X_{1,4,1,2} = X_{1,5,1,2} = 0$, hence $Y^* = 20 + 1 + 4 = 25$. Row G3-B2: $45 + 25 + 0 + 3 + 2 = 75$. Order G1-B1, G2-B1, G3-B1, G1-B2, G2-B2, G3-B2:

$$Y^* = (25;\ 22.5;\ 55;\ 25;\ 27.5;\ 75)^\top, \qquad X = (115;\ 52.5;\ 95;\ 185;\ 47.5;\ 125)^\top.$$

Step 2 — primary input (purchases from N_2), e. g. column G1-B1: $4 + 3 + 10 + 0 = 17$:

$$P^* = (17;\ 5;\ 7;\ 9;\ 11;\ 20)^\top.$$

Step 3 — coefficients and inverse (section $N_1 \times N_1$ of $\mathbf{A}$):

$$A_1 = \begin{pmatrix} 0.1739 & 0.381 & 0 & 0.1622 & 0.4211 & 0 \\ 0 & 0.2667 & 0 & 0 & 0.1263 & 0.08 \\ 0 & 0 & 0.2105 & 0 & 0 & 0.16 \\ 0 & 0.381 & 0 & 0.5405 & 0.8421 & 0 \\ 0 & 0 & 0 & 0 & 0.2105 & 0.08 \\ 0 & 0 & 0.2632 & 0 & 0 & 0.2 \end{pmatrix},$$

$$L_1 = \begin{pmatrix} 1.2105 & 0.8508 & 0.0746 & 0.4272 & 1.2375 & 0.2237 \\ 0 & 1.3636 & 0.0565 & 0 & 0.2182 & 0.1695 \\ 0 & 0 & 1.3571 & 0 & 0 & 0.2714 \\ 0 & 1.1306 & 0.1298 & 2.1765 & 2.5025 & 0.3893 \\ 0 & 0 & 0.0452 & 0 & 1.2667 & 0.1357 \\ 0 & 0 & 0.4464 & 0 & 0 & 1.3393 \end{pmatrix}.$$

Check: $L_1\, Y^* = (115;\ 52.5;\ 95;\ 185;\ 47.5;\ 125)^\top$ — identical to X from Tab 1.

Partial table N_2 (G4-B1, G5-B1, G4-B2, G5-B2):

$$Y^* = (19;\ 13;\ 49;\ 23)^\top, \qquad P^* = (26;\ 4;\ 10;\ 2.5)^\top,$$

$$A_2 = \begin{pmatrix} 0 & 0.0741 & 0.1887 & 0.3226 \\ 0.0732 & 0.037 & 0.0943 & 0.1613 \\ 0.0976 & 0 & 0 & 0 \\ 0.0732 & 0.1852 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1.0588 & 0.1518 & 0.2141 & 0.366 \\ 0.1069 & 1.087 & 0.1227 & 0.2098 \\ 0.1033 & 0.0148 & 1.0209 & 0.0357 \\ 0.0973 & 0.2124 & 0.0384 & 1.0656 \end{pmatrix},$$

Check: $L_2\, Y^* = (41;\ 27;\ 53;\ 31)^\top$.

Operating note

Operating Tab 3: *Load data from databox* (sets $N_1 = \{1, 2, 3\}$ for Examples 1 and 2) or use the data from Tab 1. After loading from the databox, tab 1 is calculated automatically; otherwise first *Calculate* in Tab 1. Assign goods to N_1 via check boxes (the rest forms N_2), *Form partial tables*. In the value representation each row (i, k) is multiplied by p_{ik} from Tab 1.

5 Tab 4 — Make-Use Matrices

5.1 Basic matrices

Use matrix U (good × sector) and make matrix V (sector × good); output of goods and sectors:

$$x_i = \sum_k V_{ki}, \qquad g_k = \sum_i V_{ki}. \tag{12}$$

Coefficient matrices:

$$D_{ki} = V_{ki}/x_i \quad \text{(market share matrix)} \tag{13}$$
$$B_{il} = U_{il}/g_l \quad \text{(use coefficients)} \tag{14}$$
$$C_{il} = V_{li}/g_l \quad \text{(production structure)} \tag{15}$$

5.2 Technology assumptions

Industry technology (uniform input structure per sector):

$$A_{ii} = B\,D \quad \text{(goods × goods)}, \qquad A_{ll} = D\,B \quad \text{(sector × sector)}. \tag{16}$$

Commodity technology (uniform input structure per good, requires $n = m$):

$$A_{ii} = B\,C^{-1}, \qquad A_{ll} = C^{-1}B. \tag{17}$$

Make-Use Example 2 (3 goods, 3 sectors)

Basic data.

$$V = \begin{pmatrix} 80 & 4 & 2 \\ 3 & 95 & 5 \\ 2 & 6 & 70 \end{pmatrix}, \quad U = \begin{pmatrix} 20 & 8 & 4 \\ 6 & 35 & 7 \\ 3 & 5 & 25 \end{pmatrix}, \quad y = (53,\ 57,\ 44)^\top.$$

According to Eq. (12): $x = (85,\ 105,\ 77)$, $g = (86,\ 103,\ 78)$. Check: row sum of $U + y = x$, e. g. $20 + 8 + 4 + 53 = 85$.

Coefficients (Eq. (13)–(15)), e. g. $D_{11} = 80/85 = 0.9412$, $B_{11} = 20/86 = 0.2326$, $C_{11} = 80/86 = 0.9302$:

$$D = \begin{pmatrix} 0.9412 & 0.0381 & 0.0260 \\ 0.0353 & 0.9048 & 0.0649 \\ 0.0235 & 0.0571 & 0.9091 \end{pmatrix}, \quad B = \begin{pmatrix} 0.2326 & 0.0777 & 0.0513 \\ 0.0698 & 0.3398 & 0.0897 \\ 0.0349 & 0.0485 & 0.3205 \end{pmatrix},$$

$$C = \begin{pmatrix} 0.9302 & 0.0291 & 0.0256 \\ 0.0465 & 0.9223 & 0.0769 \\ 0.0233 & 0.0485 & 0.8974 \end{pmatrix}.$$

Industry technology (Eq. (16)):

$$A_{ii} = \begin{pmatrix} 0.2228 & 0.0821 & 0.0577 \\ 0.0798 & 0.3152 & 0.1055 \\ 0.0421 & 0.0636 & 0.2954 \end{pmatrix}, \quad A_{ll} = \begin{pmatrix} 0.2224 & 0.0873 & 0.0600 \\ 0.0736 & 0.3133 & 0.1038 \\ 0.0412 & 0.0654 & 0.2977 \end{pmatrix},$$

$$L_{pp} = \begin{pmatrix} 1.3113 & 0.1695 & 0.1328 \\ 0.1671 & 1.5025 & 0.2386 \\ 0.0934 & 0.1457 & 1.4488 \end{pmatrix}, \quad L_{ll} = \begin{pmatrix} 1.3104 & 0.1798 & 0.1386 \\ 0.1542 & 1.4983 & 0.2347 \\ 0.0912 & 0.1500 & 1.4539 \end{pmatrix}.$$

Sector-side final demand $y' = D\,y = (53.197;\ 56.299;\ 44.504)^\top$. Exactly $L_{pp}\,y = x$ and $L_{ll}\,y' = g$ hold.

Commodity technology (Eq. (17)):

$$C^{-1} = \begin{pmatrix} 1.0773 & -0.0325 & -0.0280 \\ -0.0522 & 1.0907 & -0.0920 \\ -0.0251 & -0.0582 & 1.1200 \end{pmatrix},$$

$$A_{ii} = \begin{pmatrix} 0.2452 & 0.0742 & 0.0438 \\ 0.0552 & 0.3631 & 0.0673 \\ 0.0270 & 0.0332 & 0.3535 \end{pmatrix}, \quad A_{ll} = \begin{pmatrix} 0.2473 & 0.0713 & 0.0434 \\ 0.0607 & 0.3621 & 0.0657 \\ 0.0292 & 0.0327 & 0.3525 \end{pmatrix},$$

$$L_{pp} = \begin{pmatrix} 1.3405 & 0.1617 & 0.1076 \\ 0.1227 & 1.5936 & 0.1742 \\ 0.0623 & 0.0885 & 1.5603 \end{pmatrix}, \quad L_{ll} = \begin{pmatrix} 1.3452 & 0.1557 & 0.1059 \\ 0.1350 & 1.5915 & 0.1706 \\ 0.0674 & 0.0873 & 1.5577 \end{pmatrix}.$$

$L_{pp}\, y = x$ exactly; $L_{ll}\, y' = (85.037;\ 104.372;\ 77.824)^\top \approx g$.
Institutional/functional tables $Z = A \cdot \mathrm{diag}(x)$, goods × goods:

$$Z^{\text{industry}} = \begin{pmatrix} 18.9 & 8.6 & 4.4 \\ 6.8 & 33.1 & 8.1 \\ 3.6 & 6.7 & 22.7 \end{pmatrix}, \qquad Z^{\text{commodity}} = \begin{pmatrix} 20.8 & 7.8 & 3.4 \\ 4.7 & 38.1 & 5.2 \\ 2.3 & 3.5 & 27.2 \end{pmatrix};$$

in each case the row sum $+\,y$ yields $x = (85,\ 105,\ 77)$.

Asymmetry under commodity technology

On the goods side, L_{pp} always reproduces the vector x exactly. On the sector side, $y' = D\, y$ is formed from the market share matrix and calculated back with the matrix A_{ll} derived from C^{-1}; under commodity technology these two steps do not fit together exactly. The deviation from g (here about ± 1) is a structural property, not a numerical inaccuracy.

Operating note

Operating Tab 4: *Load data from databox* (Make-Use Example 1 or 2) or define the structure and enter U, V, y or load them as CSV. Select the technology assumption, *Calculate*. Result: D, B, C, A_{ii}, A_{ll}, Leontief inverses, institutional/functional tables.

6 Tab 5 — Derivation of the Material-Value Balance

6.1 Basic idea

As in Tab 3, the goods are split into N_1 (detailed) and N_2 (as *residual production*). The sector breakdown is retained for N_1; N_2 is condensed into one residual production column per sector.

Table A (input-output scheme variant 1): N_1 rows per sector k completely in physical units, plus for each sector l

$$\text{Res.prod.}_{ikl} = \sum_{j \in N_2} X_{ijkl}. \tag{18}$$

Table B (material-value balance): the same N_1 rows summed over the sectors k (principle of M4):

$$X_{i,jl} = \sum_k X_{ijkl}, \qquad \text{Res.prod.}_{il} = \sum_k \text{Res.prod.}_{ikl}, \qquad X_i = \sum_k X_{ik}. \tag{19}$$

Sector rows (in value terms, sum over *all* goods with prices p_{ik}):

$$\text{Sector}_k[j,l] = \sum_{i=1}^{n} p_{ik}\, X_{ijkl}. \tag{20}$$

6.2 Calculation steps for Example 1 with $N_1 = \{1,2,3\}$, prices $p_{ik} = 1$

Step 1 — Table A. Row G1-B1: deliveries to G1–G3 in sectors 1 and 2 unchanged (20, 20, 0 | 30, 20, 0); residual production sector $1 = X_{1,4,1,1} + X_{1,5,1,1} = 1 + 4 = 5$, sector $2 = 0$. Row G2-B1: residual production sector $2 = 2 + 0.5 = 2.5$. Row G3-B2: residual production sector $1 = 25 + 0 = 25$, sector $2 = 3 + 2 = 5$. Sector row 1, column G1-B1 according to Eq. (20): $1 \cdot 20 + 1 \cdot 0 + 1 \cdot 0 + 1 \cdot 4 + 1 \cdot 3 = 27$.

	G1B1	G2B1	G3B1	G1B2	G2B2	G3B2	Res.B1	Res.B2	Y	X
G1-B1	20	20	0	30	20	0	5	0	20	115
G2-B1	0	14	0	0	6	10	0	2.5	20	52.5
G3-B1	0	0	20	0	0	20	0	5	50	95
G1-B2	0	20	0	100	40	0	0	0	25	185
G2-B2	0	0	0	0	10	10	0	0	27.5	47.5
G3-B2	0	0	25	0	0	25	25	5	45	125
Sector 1	27	34	20	30	31	40	11	37.5	100	330.5
Sector 2	10	25	32	109	56	45	37	5	122.5	441.5

Step 2 — Table B (rows summed over the sectors), e. g. good 1 to G2-B1: $20 + 20 = 40$, to G1-B2: $30 + 100 = 130$, $X_1 = 115 + 185 = 300$:

	G1B1	G2B1	G3B1	G1B2	G2B2	G3B2	Res.B1	Res.B2	Y	X
G1	20	40	0	130	60	0	5	0	45	300
G2	0	14	0	0	16	20	0	2.5	47.5	100
G3	0	0	45	0	0	45	25	10	95	220
Sector 1	27	34	20	30	31	40	11	37.5	100	330.5
Sector 2	10	25	32	109	56	45	37	5	122.5	441.5

With $p_{ik} = 1$ the sums of the sector rows are 330.5 and 441.5 (= sector output from M9).

Operating note

Operating Tab 5: *Load data from databox* (sets $N_1 = \{1,2,3\}$ for Examples 1 and 2), *Take data from tab 1* or own data/CSV. Enter prices p_{ik} per good and sector or load them as CSV (default 1; template with header row G1-B1; G2-B1; . . . ; the columns are assigned on the basis of the labels, older files with one price per good apply to all sectors), *Calculate*, check N_1, *Create tables* — yields Table A, Table A (valued), Table B as well as coefficient matrices and vectors.

7 Databox

- **Loading:** In tabs 1, 3, 4 and 5, *Load data from databox* opens a list of the matching datasets — interindustry tables for tabs 1, 3 and 5, make-use datasets for Tab 4. Static datasets are listed under "More datasets" and are loaded as n goods with one sector ($Z = A\,\mathrm{diag}(X)$, $X = (I - A)^{-1}Y$).
- **Saving** (Tab 1): with the complete structure (n, m, Z, Y) in the field `io4d`, plus A and Y of dimension nm; tag "Four-dimensional".
- **Export** (databox management): CSV in the format of the IO calculator (can be re-imported), Excel with the sheets Z or U/V.
- The English tools use their own databox (separate from the German version).

A Glossary of Coefficients and Quantities

Symbol	Meaning
X_{ijkl}	delivery of good i from sector k for the production of good j in sector l
X_{ik}	gross output of good i in sector k
Y_{ik}	final demand for good i from sector k
a_{ijkl}	input coefficient, $= X_{ijkl}/X_{jl}$ (Eq. 2)
A, L	coefficient matrix and Leontief inverse of the four-dimensional model
$\rho(A)$	spectral radius; $\rho < 1$ ensures solvability even with column sums above 1 (physical units)
w_{ijkl}	weight matrix: share of the supplying sector k in the delivery of good i to use (j, l)
p_{ik}	price of good i from sector k
A^p, X^p	price-valued coefficient matrix and gross output
M1–M9	model variants depending on the classification characteristic retained (Tab 2)
d_{jl}	share of sector l in the production of good j, $= X_{jl}/X_j$.
N_1, N_2	disjoint subsets of the goods (tabs 3 and 5)
Y^*_{ik}	modified final demand of the partial table (Eq. 9)
P^*_{jl}	primary input of the partial table from the other group (Eq. 10)
U_{il}, V_{ki}	use and make matrix (Tab 4)
x_i, g_k	output of goods and sectors (Eq. 12)
D, B, C	market share, use and production structure matrix (Tab 4)
A_{ii}, A_{ll}	coefficient matrix goods×goods or sector×sector under industry or commodity technology
Table A	input-output scheme variant 1: N_1 rows per sector plus residual production (Tab 5)
Table B	material-value balance: Table A summed over the sectors (Tab 5)
Res.prod.$_{ikl}$	delivery of $(i, k) \in N_1$ to N_2 in sector l (Eq. 18)
Sector$_k$	sector row in value terms, sum over all goods with price weighting (Eq. 20)

Literature

P. Fleissner, W. Böhme, H.-U. Brautzsch, J. Höhne, J. Siassi, K. Stark: *Input-Output-Analyse. Eine Einführung in Theorie und Anwendungen* [Input-output analysis. An introduction to theory and applications]. Springer-Verlag, Wien, New York 1993.