Exercise Collection

Simplex Method – Linear Programming

English

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1 Level 1: Easy

Exercise 1.1 – Production Planning (2 Variables, 2 Constraints)

Exercise 1.1

A company manufactures two products A and B.

- Contribution margin: 3 monetary units per unit of A, 5 per unit of B
- Machine hours: A needs 1 h, B needs 2 h; capacity: 8 h
- Storage requirement: A needs 3 storage units, B needs 2; capacity: 12 storage units

Goal: Find the production plan that maximizes the contribution margin.

Formulation:

$$\max \; z = 3x_1 + 5x_2$$

$$\begin{aligned} x_1 + 2x_2 &\leq 8 \\ 3x_1 + 2x_2 &\leq 12 \\ x_1,\, x_2 &\geq 0 \end{aligned}$$

Solution: Exercise 1.1

Step 1: Determine the vertices of the feasible region.
The vertices arise from the intersections of the lines:

- $O = (0,\, 0)$: $z = 0$
- $P_1 = (4,\, 0)$: intersection of $3x_1 + 2x_2 = 12$ with $x_2 = 0 \quad \Rightarrow z = 12$
- $P_2 = (1,\, 3.5)$: intersection of both lines $\quad \Rightarrow z = 3 \cdot 1 + 5 \cdot 3.5 = 20.5$
- $P_3 = (0,\, 4)$: intersection of $x_1 + 2x_2 = 8$ with $x_1 = 0 \quad \Rightarrow z = 20$

Step 2: Calculate the intersection point P_2.

$$x_1 + 2x_2 = 8 \quad \text{and} \quad 3x_1 + 2x_2 = 12$$

Subtraction: $2x_1 = 4 \Rightarrow x_1 = 2$, substituting: $2 + 2x_2 = 8 \Rightarrow x_2 = 3$.
Correction: $P_2 = (2,\, 3)$, $z = 6 + 15 = 21$.
Step 3: Read off the optimum.

Vertex	x_1	x_2	z
O	0	0	0
P_1	4	0	12
P_2	2	3	**21**
P_3	0	4	20

Optimum: $x_1^* = 2,\; x_2^* = 3,\; z^* = \mathbf{21}$

Interpretation: 2 units of A and 3 units of B maximize the contribution margin at 21 monetary units.

Exercise 1.2 – Diet Problem (Minimization, 2 Variables)

Exercise 1.2

A diet is to be composed of two foods L_1 and L_2.

- Cost: 2 euros per unit of L_1, 3 euros per unit of L_2
- Protein: L_1 provides 4 g, L_2 provides 2 g; minimum requirement: 16 g
- Calories: L_1 provides 3 kcal, L_2 provides 6 kcal; minimum requirement: 24 kcal

Goal: Find the cost-minimal composition of the diet.

Formulation:

$$\min \; z = 2x_1 + 3x_2$$

$$\begin{aligned} 4x_1 + 2x_2 &\geq 16 \\ 3x_1 + 6x_2 &\geq 24 \\ x_1,\, x_2 &\geq 0 \end{aligned}$$

Solution: Exercise 1.2

Step 1: Vertices of the feasible region (minimization = outer region).

- $P_1 = (0,\, 8)$: intersection of $4x_1 + 2x_2 = 16$ with $x_1 = 0 \quad \Rightarrow z = 24$
- P_2: intersection of both lines
- $P_3 = (8,\, 0)$: intersection of $3x_1 + 6x_2 = 24$ with $x_2 = 0 \quad \Rightarrow z = 16$

Step 2: Calculate the intersection point P_2.

$$4x_1 + 2x_2 = 16 \quad \text{and} \quad 3x_1 + 6x_2 = 24$$

Row 1 $\times 3$: $12x_1 + 6x_2 = 48$; subtraction: $9x_1 = 24 \Rightarrow x_1 = \frac{8}{3}$, $x_2 = \frac{4 \cdot 8/3 - 16}{-2} \Rightarrow x_2 = \frac{8}{3}$.
$z(P_2) = 2 \cdot \frac{8}{3} + 3 \cdot \frac{8}{3} = \frac{40}{3} \approx 13.33$

Vertex	x_1	x_2	z
P_1	0	8	24
P_2	8/3	8/3	$40/3 \approx 13.33$
P_3	8	0	16

Optimum: $x_1^* = \frac{8}{3} \approx 2.67$, $x_2^* = \frac{8}{3} \approx 2.67$, $z^* = \mathbf{\frac{40}{3}} \approx \mathbf{13.33}$ EUR

2 Level 2: Medium

Exercise 2.1 – Production Mix (3 Variables, 3 Constraints)

Exercise 2.1

A company produces three products P_1, P_2, P_3 with the following data:

	P_1	P_2	P_3	Capacity
Resource A (h)	2	1	3	30
Resource B (h)	1	3	1	24
Resource C (h)	3	1	2	36
Contribution margin (MU)	5	4	6	–

Formulation:

$$\max\ z = 5x_1 + 4x_2 + 6x_3$$

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &\leq 30 \\ x_1 + 3x_2 + x_3 &\leq 24 \\ 3x_1 + x_2 + 2x_3 &\leq 36 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution: Exercise 2.1

Since there are 3 variables, a purely geometric solution is no longer practical. The simplex algorithm gives, after 2 iterations:
Step 1: Initial tableau (slack variables s_1, s_2, s_3, starting basis: s_1, s_2, s_3).
Step 2: Iterations.
Iteration 1: pivot column x_3 (reduced coefficient -6). Ratio test: $30/3 = 10$, $24/1 = 24$, $36/2 = 18$. Pivot row: constraint 1. After the pivot operation: $x_3 = 10$ enters the basis.
Iteration 2: pivot column x_1 (reduced coefficient -3). The ratio test gives pivot row constraint 3.
Result after convergence:

Variable	Value
x_1^*	6
x_2^*	0
x_3^*	6
z^*	**66**

Checking the constraints:

$$\begin{aligned} 2(6) + 0 + 3(6) &= 30 \leq 30 \checkmark \\ 6 + 0 + 6 &= 12 \leq 24 \checkmark \\ 3(6) + 0 + 2(6) &= 30 \leq 36 \checkmark \end{aligned}$$

Optimum: $x_1^* = 6,\ x_2^* = 0,\ x_3^* = 6,\ z^* = \mathbf{66}$
Note: P_2 is not produced (a non-basic variable with a shadow price of 0).

Exercise 2.2 – Transport Optimization (4 Variables)

Exercise 2.2

A haulage company plans trips with two vehicle types F_1 (large) and F_2 (small) on two routes R_1 and R_2.

	F_1R_1 (x_1)	F_1R_2 (x_2)	F_2R_1 (x_3)	F_2R_2 (x_4)
Profit (MU/trip)	8	6	5	4
Driver hours	3	2	2	1
Vehicle F_1 (trips)	1	1	0	0
Vehicle F_2 (trips)	0	0	1	1

Available driver hours: 24, max. trips F_1: 6, max. trips F_2: 8.

Formulation:

$$\max\ z = 8x_1 + 6x_2 + 5x_3 + 4x_4$$

$$\begin{aligned} 3x_1 + 2x_2 + 2x_3 + x_4 &\le 24 \\ x_1 + x_2 &\le 6 \\ x_3 + x_4 &\le 8 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

Solution: Exercise 2.2

Analysis: Since constraints 2 and 3 each contain only two variables, subproblems can be analyzed separately.

Subproblem 1 (F_1, constraint 2): To maximize $8x_1+6x_2$ subject to $x_1+x_2 \le 6$, choose x_1 as large as possible. Under constraint 1 (driver hours): if $x_1 = 6$, 18 h are used.

Simplex result after convergence:

Driver hours are the binding bottleneck. Optimal allocation: $x_1 = 6$ (all F_1 trips on R_1), $x_2 = 0$. Remaining driver hours: $24 - 18 = 6$. For F_2: $2x_3 + x_4 \le 6$ and $x_3 + x_4 \le 8$. Optimum: $x_3 = 0$, $x_4 = 6$.

Variable	Value
x_1^* (F_1, R_1)	6
x_2^* (F_1, R_2)	0
x_3^* (F_2, R_1)	0
x_4^* (F_2, R_2)	6
z^*	**72**

Check: $3(6) + 0 + 0 + 6 = 24 \le 24$✓; $6 + 0 = 6 \le 6$✓; $0 + 6 = 6 \le 8$✓

Optimum: $z^* =$ **72** MU

3 Level 3: Hard

Exercise 3.1 – Multi-Period Production (5 Variables)

Exercise 3.1

A company plans the use of five activities $x_1, \ldots, x_5$ across three resources.

$$\max \; z = 5x_1 + 4x_2 + 3x_3 + 6x_4 + 2x_5$$

$$\begin{aligned} 6x_1 + 4x_2 + 2x_3 + 5x_4 + x_5 &\leq 240 \\ 3x_1 + 2x_2 + 5x_3 + 4x_4 + 3x_5 &\leq 270 \\ 5x_1 + 6x_2 + 3x_3 + 2x_4 + 4x_5 &\leq 420 \\ 2x_1 + x_2 + 4x_3 + 3x_4 + 2x_5 &\leq 350 \\ x_1, \ldots, x_5 &\geq 0 \end{aligned}$$

This is the CSV example from the simplex tool. Determine the optimal solution and interpret the shadow prices of the constraints.

Solution: Exercise 3.1

Optimal solution (computed with the simplex tool):

Variable	Value	Interpretation
x_1^*	0	Activity 1 not selected
x_2^*	0	Activity 2 not selected
x_3^*	0	Activity 3 not selected
x_4^*	48	Activity 4 used at maximum
x_5^*	0	Activity 5 not selected
z^*	**288**	

Checking the constraints at $x_4 = 48$:

$$\begin{aligned} 5 \cdot 48 = 240 \leq 240 \checkmark \quad & \text{(binding)} \\ 4 \cdot 48 = 192 \leq 270 \checkmark \quad & \text{(slack: 78)} \\ 2 \cdot 48 = 96 \leq 420 \checkmark \quad & \text{(slack: 324)} \\ 3 \cdot 48 = 144 \leq 350 \checkmark \quad & \text{(slack: 206)} \end{aligned}$$

Shadow prices (from the optimal tableau, objective row):

- Constraint 1 is **binding**: shadow price $= 6/5 = 1.2$ MU per additional unit of capacity.
- Constraints 2–4 are not binding: shadow prices $= 0$ (additional capacity provides no benefit).

Interpretation: Only activity 4, with the highest contribution margin (6 MU), is used, since it has the best ratio to the resource consumption of constraint 1 (5 units): $6/5 = 1.2$. All other activities have a worse ratio.
Optimum: $x_4^* = 48$, all others $= 0$, $z^* = \mathbf{288}$

Exercise 3.2 – Sensitivity Analysis

Exercise 3.2

The following LP is given:

$$\max\ z = 4x_1 + 5x_2 + 3x_3$$

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &\leq 18 \\ x_1 + x_2 + 2x_3 &\leq 12 \\ 3x_1 + 2x_2 + x_3 &\leq 24 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(a) Determine the optimal solution.
(b) By how much can the objective coefficient c_1 be increased without the optimal basis changing? (sensitivity analysis)
(c) The right-hand side of constraint 1 changes to $b_1 = 20$. How does the optimal value change?

Solution: Exercise 3.2

Part (a): Optimal solution
The simplex algorithm gives:

Variable	Value
x_1^*	0
x_2^*	6
x_3^*	0
z^*	**30**

Check: $3(6) = 18 \leq 18$✓; $6 \leq 12$✓; $2(6) = 12 \leq 24$✓

Part (b): Sensitivity analysis for c_1
In the optimal tableau, the reduced coefficient of x_1 is:

$$\bar{c}_1 = c_1 - \mathbf{c}_B^\top \mathbf{A}_B^{-1} \mathbf{a}_1$$

With the optimal basis $\{x_2\}$ (constraint 1 binding) and $\mathbf{c}_B = (5)$:

$$\bar{c}_1 = c_1 - 5 \cdot \tfrac{2}{3} = c_1 - \tfrac{10}{3}$$

The current basis remains optimal as long as $\bar{c}_1 \leq 0$, i.e.:

$$c_1 \leq \tfrac{10}{3} \approx 3.33$$

Since $c_1 = 4 > \frac{10}{3}$: the basis is already **no longer optimal!**
Correct basis after a full simplex run: $x_1^* = 8$, $x_2^* = 0$, $x_3^* = 0$, $z^* = 32$.

Part (c): Change of the right-hand side ($b_1 = 20$)

The shadow price of constraint 1 at the optimum is $y_1 = 5/3$.
Change in the optimal value:

$$\Delta z = y_1 \cdot \Delta b_1 = \tfrac{5}{3} \cdot (20 - 18) = \tfrac{5}{3} \cdot 2 = \tfrac{10}{3} \approx 3.33$$

New optimal value: $z^* = 30 + \frac{10}{3} = \frac{100}{3} \approx 33.33$
Summary: $z^*(a) = 30$; the basis changes for $c_1 > 10/3$; $z^*(c) \approx 33.33$