# Exercise Collection

## Transportation Problem

English

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25 October 2026

The documentation on this website was developed with the support of the AI Claude (Anthropic).

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# 1 Level 1: Easy

## Exercise 1.1 – Northwest Corner Method

**Exercise 1.1**

The following transportation problem is given (cost per unit):

| | $N_1$ | $N_2$ | $N_3$ | Supply |
|---|---|---|---|---|
| $A_1$ | 2 | 3 | 5 | 30 |
| $A_2$ | 4 | 1 | 6 | 40 |
| $A_3$ | 3 | 5 | 2 | 30 |
| Demand | 25 | 35 | 40 | 100 |

(a) Check whether the problem is balanced.

(b) Determine a starting solution using the Northwest Corner Method.

(c) Calculate the total cost of the starting solution.

(d) How many basic variables does a non-degenerate solution have?

**Solution: Exercise 1.1**

**(a) Balance check:** $\sum a_i = 30 + 40 + 30 = 100 = 25 + 35 + 40 = \sum b_j$ ✓

**(b) Northwest Corner Method:**

1. $(1,1)$: $\min(30, 25) = 25$, $x_{11} = 25$. $a_1 = 5$, $b_1 = 0$.
2. $(1,2)$: $\min(5, 35) = 5$, $x_{12} = 5$. $a_1 = 0$, $b_2 = 30$.
3. $(2,2)$: $\min(40, 30) = 30$, $x_{22} = 30$. $a_2 = 10$, $b_2 = 0$.
4. $(2,3)$: $\min(10, 40) = 10$, $x_{23} = 10$. $a_2 = 0$, $b_3 = 30$.
5. $(3,3)$: $\min(30, 30) = 30$, $x_{33} = 30$. Done.

| | $N_1$ | $N_2$ | $N_3$ | Supply |
|---|---|---|---|---|
| $A_1$ | **25**$_{(2)}$ | **5**$_{(3)}$ | $_{(5)}$ | 30 |
| $A_2$ | $_{(4)}$ | **30**$_{(1)}$ | **10**$_{(6)}$ | 40 |
| $A_3$ | $_{(3)}$ | $_{(5)}$ | **30**$_{(2)}$ | 30 |
| Demand | 25 | 35 | 40 | |

**(c) Total cost:** $K = 25{\cdot}2 + 5{\cdot}3 + 30{\cdot}1 + 10{\cdot}6 + 30{\cdot}2 = 50 + 15 + 30 + 60 + 60 = \mathbf{215}$

**(d) Basic variables:** $m + n - 1 = 3 + 3 - 1 = 5$

## Exercise 1.2 – Minimum Cost Method

**Exercise 1.2**

Use the same cost matrix and the same supply/demand quantities as in Exercise 1.1.

(a) Determine a starting solution using the Minimum Cost Method.

(b) Calculate the total cost and compare it with the Northwest Corner solution.

(c) Which cell has the lowest cost and receives the first allocation?

**Solution: Exercise 1.2**

**(a) Minimum Cost Method:**
Cheapest cells in order: $c_{22} = 1$, $c_{11} = 2$, $c_{33} = 2$, $c_{12} = 3$, $c_{31} = 3$, ...

1. $c_{22} = 1$: $x_{22} = \min(40, 35) = 35$. $b_2 = 0$, $a_2 = 5$.
2. $c_{11} = 2$: $x_{11} = \min(30, 25) = 25$. $b_1 = 0$, $a_1 = 5$.
3. $c_{33} = 2$: $x_{33} = \min(30, 40) = 30$. $a_3 = 0$, $b_3 = 10$.
4. $c_{12} = 3$ blocked (column 2 exhausted). $c_{13} = 5$: $x_{13} = \min(5, 10) = 5$. $a_1 = 0$.
5. $c_{23} = 6$: $x_{23} = \min(5, 5) = 5$. Done.

| | $N_1$ | $N_2$ | $N_3$ | Supply |
|---|---|---|---|---|
| $A_1$ | **25**$_{(2)}$ | $_{(3)}$ | **5**$_{(5)}$ | 30 |
| $A_2$ | $_{(4)}$ | **35**$_{(1)}$ | **5**$_{(6)}$ | 40 |
| $A_3$ | $_{(3)}$ | $_{(5)}$ | **30**$_{(2)}$ | 30 |
| Demand | 25 | 35 | 40 | |

**(b) Total cost:** $K = 25{\cdot}2 + 5{\cdot}5 + 35{\cdot}1 + 5{\cdot}6 + 30{\cdot}2 = 50 + 25 + 35 + 30 + 60 = \mathbf{200}$
Comparison: NW = 215, MC = 200 $\Rightarrow$ MC is better by 15.
**(c) First allocation:** Cell $(2, 2)$ with $c_{22} = 1$.
**Minimum-cost solution:** $K = 200$ **(better than NW with 215)**

# 2 Level 2: Medium

## Exercise 2.1 – Vogel's Approximation Method

**Exercise 2.1**

The following transportation problem is given:

| | $N_1$ | $N_2$ | $N_3$ | $N_4$ | Supply |
|---|---|---|---|---|---|
| $A_1$ | 4 | 8 | 1 | 6 | 120 |
| $A_2$ | 7 | 2 | 5 | 3 | 80 |
| $A_3$ | 3 | 6 | 4 | 9 | 100 |
| Demand | 90 | 60 | 70 | 80 | 300 |

(a) Calculate all penalty costs for the first iteration of Vogel's Approximation Method.

(b) Perform the complete Vogel's Approximation Method.

(c) Calculate the total cost.

**Solution: Exercise 2.1**

**(a) Penalty costs, iteration 1:**

| | $N_1$ | $N_2$ | $N_3$ | $N_4$ | Supply | **Penalty** |
|---|---|---|---|---|---|---|
| $A_1$ | 4 | 8 | 1 | 6 | 120 | $4-1=\mathbf{3}$ |
| $A_2$ | 7 | 2 | 5 | 3 | 80 | $3-2=\mathbf{1}$ |
| $A_3$ | 3 | 6 | 4 | 9 | 100 | $4-3=\mathbf{1}$ |
| Demand | 90 | 60 | 70 | 80 | | |
| **Penalty** | $4-3=1$ | $6-2=4$ | $4-1=\mathbf{3}$ | $6-3=3$ | | |

Highest penalty: column $N_2$ (penalty 4). Cheapest cell in $N_2$: $(2,2)$ with $c=2$. $x_{22}=\min(80,60)=60$. $a_2=20$, $b_2=0$.

**(b) Complete VAM:**

*Iteration 2* (without $N_2$): recompute penalty costs. Highest penalty: $N_3$ or $A_1$ (penalty 3). Choose $N_3$: cheapest cell $(1,3)$, $c=1$. $x_{13}=\min(120,70)=70$. $a_1=50$.

*Iteration 3*: highest penalty: $A_1$ (penalty $6-4=2$). $N_1$: cheapest cell $(1,1)$, $c=4$. $x_{11}=\min(50,90)=50$. $a_1=0$, $b_1=40$.

*Iteration 4*: $A_2$ still open ($a_2=20$), $A_3$ still open. Compute penalties. Cheapest: $(2,4)$, $c=3$. $x_{24}=\min(20,80)=20$. $a_2=0$, $b_4=60$.

*Iteration 5*: $A_3$: $x_{31}=\min(100,40)=40$. $b_1=0$, $a_3=60$.

*Iteration 6*: $x_{34}=60$. Done.

**(c) Total cost:** $K = 50 \cdot 4 + 70 \cdot 1 + 60 \cdot 2 + 20 \cdot 3 + 40 \cdot 3 + 60 \cdot 9 = 200 + 70 + 120 + 60 + 120 + 540 = \mathbf{1110}$
**VAM solution:** $K = 1110$

## Exercise 2.2 – One MODI Iteration

**Exercise 2.2**

The following basic solution of a $3 \times 3$ transportation problem is given:

| | $N_1$ | $N_2$ | $N_3$ | Supply |
|---|---|---|---|---|
| $A_1$ | **40**$_{(2)}$ | **30**$_{(3)}$ | $_{(5)}$ | 70 |
| $A_2$ | $_{(7)}$ | **10**$_{(4)}$ | **30**$_{(1)}$ | 40 |
| $A_3$ | $_{(3)}$ | $_{(2)}$ | **50**$_{(6)}$ | 50 |
| Demand | 40 | 40 | 80 | |

(a) Calculate the potentials $u_i$ and $v_j$ (set $u_1 = 0$).

(b) Calculate all reduced costs $\delta_{ij}$ for non-basic cells.

(c) Is the solution optimal? If not: which cell enters?

(d) Perform one MODI iteration (loop, $\theta$, new solution, new cost).

**Solution: Exercise 2.2**

**(a) Potentials** ($u_1 = 0$):

$$x_{11} : c_{11} = u_1 + v_1 \Rightarrow 2 = 0 + v_1 \Rightarrow v_1 = 2$$
$$x_{12} : c_{12} = u_1 + v_2 \Rightarrow 3 = 0 + v_2 \Rightarrow v_2 = 3$$
$$x_{22} : c_{22} = u_2 + v_2 \Rightarrow 4 = u_2 + 3 \Rightarrow u_2 = 1$$
$$x_{23} : c_{23} = u_2 + v_3 \Rightarrow 1 = 1 + v_3 \Rightarrow v_3 = 0$$
$$x_{33} : c_{33} = u_3 + v_3 \Rightarrow 6 = u_3 + 0 \Rightarrow u_3 = 6$$

**(b) Reduced costs** $\delta_{ij} = c_{ij} - u_i - v_j$:

$$\delta_{13} = 5 - 0 - 0 = 5$$
$$\delta_{21} = 7 - 1 - 2 = 4$$
$$\delta_{31} = 3 - 6 - 2 = \mathbf{-5}$$
$$\delta_{32} = 2 - 6 - 3 = \mathbf{-7}$$

**(c) Optimality:** No – $\delta_{32} = -7 < 0$. Entering cell: $(3, 2)$.
**(d) MODI iteration:**
Loop for $(3, 2)$: $(3, 2)^+ \to (3, 3)^- \to (2, 3)^+ \to (2, 2)^- \to (3, 2)$... Check: $(3, 2) \to (1, 2) \to (1, 1) \to (3, 1)$? No, $(3, 1)$ is not a basic variable.

Alternative attempt: $(3,2)^+ \to (1,2)^- \to (1,1)^+$? – $(1,1)$ is not in column 2.
Correct loop: $(3,2)^+ \to (2,2)^- \to (2,3)^+ \to (3,3)^-$.
$(-)$-positions: $x_{22} = 10$, $x_{33} = 50$. $\theta = \min(10, 50) = 10$.
Update: $x_{32} = 0 + 10 = 10$, $x_{22} = 10 - 10 = 0$ (leaves), $x_{23} = 30 + 10 = 40$, $x_{33} = 50 - 10 = 40$.
Cost reduction: $\Delta K = \delta_{32} \cdot \theta = (-7) \cdot 10 = -70$.
Old cost: $40 \cdot 2 + 30 \cdot 3 + 10 \cdot 4 + 30 \cdot 1 + 50 \cdot 6 = 80 + 90 + 40 + 30 + 300 = 540$.
New cost: $540 - 70 = \mathbf{470}$.
**New basic solution with $K = 470$. Entering cell $(3,2)$, leaving cell $(2,2)$.**

# 3 Level 3: Hard

## Exercise 3.1 – Complete Optimization

**Exercise 3.1**

The following transportation problem is given:

| | $N_1$ | $N_2$ | $N_3$ | $N_4$ | Supply |
|---|---|---|---|---|---|
| $A_1$ | 2 | 3 | 1 | 5 | 120 |
| $A_2$ | 7 | 3 | 4 | 6 | 80 |
| $A_3$ | 5 | 4 | 7 | 3 | 100 |
| Demand | 90 | 70 | 80 | 60 | 300 |

(a) Determine a starting solution with the Northwest Corner Method and calculate the cost.

(b) Perform as many MODI iterations as needed to reach the optimal solution.

(c) Show that the final solution is optimal (all $\delta_{ij} \geq 0$).

(d) Calculate the cost reduction relative to the starting solution, in percent.

**Solution: Exercise 3.1**

**(a) Northwest Corner starting solution:**
$(1,1)$: $x_{11} = 90$, $a_1 = 30$. $(1,2)$: $x_{12} = 30$, $a_1 = 0$. $(2,2)$: $x_{22} = 40$, $b_2 = 0$. $(2,3)$: $x_{23} = 40$, $a_2 = 0$. $(3,3)$: $x_{33} = 40$, $b_3 = 0$. $(3,4)$: $x_{34} = 60$.
$K_0 = 90{\cdot}2{+}30{\cdot}3{+}40{\cdot}3{+}40{\cdot}4{+}40{\cdot}7{+}60{\cdot}3 = 180{+}90{+}120{+}160{+}280{+}180 = \mathbf{1010}$
**(b) MODI iteration 1:**
Potentials: $u_1 = 0$, $v_1 = 2$, $v_2 = 3$, $u_2 = 0$, $v_3 = 4$, $u_3 = 3$, $v_4 = 0$.
Reduced costs: $\delta_{13} = 1{-}0{-}4 = -3$, $\delta_{14} = 5$, $\delta_{21} = 5$, $\delta_{24} = 6$, $\delta_{31} = 0$, $\delta_{32} = -2$.
Entering: $(1,3)$ with $\delta = -3$.
Loop: $(1,3)^+ \to (1,2)^- \to (2,2)^+ \to (2,3)^-$. $\theta = \min(30, 40) = 30$.
$x_{13} = 30$, $x_{12} = 0$ (leaves), $x_{22} = 70$, $x_{23} = 10$. $\Delta K = -3 \cdot 30 = -90$. $K_1 = \mathbf{920}$.
**MODI iteration 2:**
New basis: $x_{11} = 90, x_{13} = 30, x_{22} = 70, x_{23} = 10, x_{33} = 40, x_{34} = 60$.
Potentials: $u_1 = 0$, $v_1 = 2$, $v_3 = 1$, $u_2 = 3$, $v_2 = 0$, $u_3 = 6$, $v_4 = -3$.
Reduced costs (selection): $\delta_{12} = 3$, $\delta_{14} = 8$, $\delta_{21} = 2$, $\delta_{24} = 6$, $\delta_{31} = -3$, $\delta_{32} = -2$.
Entering: $(3,1)$ with $\delta = -3$.
Loop: $(3,1)^+ \to (3,3)^- \to (1,3)^+ \to (1,1)^-$. $\theta = \min(40, 90) = 40$.
$x_{31} = 40$, $x_{33} = 0$ (leaves), $x_{13} = 70$, $x_{11} = 50$. $\Delta K = -3{\cdot}40 = -120$. $K_2 = \mathbf{800}$.
**(c) Checking optimality:**
New basis: $x_{11} = 50, x_{13} = 70, x_{22} = 70, x_{23} = 10, x_{31} = 40, x_{34} = 60$.
Potentials: $u_1 = 0$, $v_1 = 2$, $v_3 = 1$, $u_2 = 3$, $v_2 = 0$, $u_3 = 3$, $v_4 = 0$.

All reduced costs:

$$\delta_{12} = 3-0-0 = 3 \geq 0 \qquad \delta_{14} = 5-0-0 = 5 \geq 0$$
$$\delta_{21} = 7-3-2 = 2 \geq 0 \qquad \delta_{24} = 6-3-0 = 3 \geq 0$$
$$\delta_{32} = 4-3-0 = 1 \geq 0 \qquad \delta_{33} = 7-3-1 = 3 \geq 0$$

All $\delta_{ij} \geq 0 \Rightarrow$ **optimal solution reached!**
Optimal solution:

| | $N_1$ | $N_2$ | $N_3$ | $N_4$ | Supply |
|---|---|---|---|---|---|
| $A_1$ | **50**$_{(2)}$ | $_{(3)}$ | **70**$_{(1)}$ | $_{(5)}$ | 120 |
| $A_2$ | $_{(7)}$ | **70**$_{(3)}$ | **10**$_{(4)}$ | $_{(6)}$ | 80 |
| $A_3$ | **40**$_{(5)}$ | $_{(4)}$ | $_{(7)}$ | **60**$_{(3)}$ | 100 |
| Demand | 90 | 70 | 80 | 60 | |

$K^* = 50{\cdot}2+70{\cdot}1+70{\cdot}3+10{\cdot}4+40{\cdot}5+60{\cdot}3 = 100+70+210+40+200+180 =$ **800**
**(d) Cost reduction:** $\frac{1010-800}{1010} \cdot 100\% = \frac{210}{1010} \cdot 100\% \approx$ **20.8%**

## Exercise 3.2 – Unbalanced Problem and Degeneracy

**Exercise 3.2**

The following **unbalanced** transportation problem is given:

| | $N_1$ | $N_2$ | $N_3$ | Supply |
|---|---|---|---|---|
| $A_1$ | 3 | 5 | 2 | 80 |
| $A_2$ | 6 | 1 | 4 | 70 |
| $A_3$ | 4 | 3 | 7 | 60 |
| Demand | 50 | 90 | 40 | |

(a) Determine whether supply or demand is larger, and add a dummy supplier or demander with cost 0.

(b) Determine a starting solution using the Minimum Cost Method.

(c) Explain what **degeneracy** means, and describe when it occurs.

(d) Does degeneracy occur in the starting solution? Justify.

(e) Perform one MODI iteration and calculate the optimal cost (excluding dummy transport quantities).

**Solution: Exercise 3.2**

**(a) Balance check:** $\sum a_i = 80 + 70 + 60 = 210$, $\sum b_j = 50 + 90 + 40 = 180$. Supply $>$ demand: surplus of 30. Add a dummy demander $N_4$ with $b_4 = 30$ and $c_{i4} = 0$ for all $i$.

**(b) Minimum Cost Method** (with $N_4$):

Cheapest cells: $c_{22} = 1$, $c_{13} = 2$, $c_{23} = 4$ would come after $c_{31} = 4$... Order: $(2,2) = 1$, $(1,3) = 2$, $(3,1) = 4$ would come after the dummy column $c_{i4} = 0$.

With $c_{i4} = 0$: first $(1,4)$, $(2,4)$ or $(3,4)$ – choose $(1,4)$: $x_{14} = \min(80, 30) = 30$. $b_4 = 0$.

Then $(2,2) = 1$: $x_{22} = \min(70, 90) = 70$. $a_2 = 0$, $b_2 = 20$. Then $(1,3) = 2$: $x_{13} = \min(50, 40) = 40$. $b_3 = 0$, $a_1 = 10$. Then $(1,2) = 5$ or $(3,2) = 3$: $(3,2) = 3$: $x_{32} = \min(60, 20) = 20$. $b_2 = 0$, $a_3 = 40$. Then $(1,2) = 5$: $x_{12} = \min(10, 0)$... $b_2 = 0$ exhausted. $(1,1) = 3$: $x_{11} = 10$. $a_1 = 0$. Finally $x_{31} = \min(40, 40) = 40$. Done.

Starting solution: $x_{14} = 30$, $x_{11} = 10$, $x_{13} = 40$, $x_{22} = 70$, $x_{32} = 20$, $x_{31} = 40$.

Cost (excluding $N_4$): $10{\cdot}3{+}40{\cdot}2{+}70{\cdot}1{+}40{\cdot}4{+}20{\cdot}3 = 30{+}80{+}70{+}160{+}60 =$ **400**

**(c) Degeneracy:** A basic solution is **degenerate** if the number of basic variables is smaller than $m + n - 1$, i.e., a basic variable takes the value 0. This occurs when an allocation step simultaneously exhausts the supply and demand of a row/column. Degeneracy requires artificially introducing a basic variable with value 0 (an "epsilon allocation").

**(d) Degeneracy in the starting solution:** The extended problem has $m = 3$, $n = 4$, so $m + n - 1 = 6$ basic variables are required. The solution above has 6 basic variables $(x_{14}, x_{11}, x_{13}, x_{22}, x_{32}, x_{31}) \Rightarrow$ **no degeneracy**.

**(e) MODI iteration:**

Basis (extended problem): calculate potentials, check reduced costs. $u_1 = 0$: $v_1 = 3$ (from $x_{11}$), $v_3 = 2$ (from $x_{13}$), $v_4 = 0$ (from $x_{14}$). $u_2$: from $x_{22}$: $u_2 = 1{-}v_2$... $v_2$ still unknown. From $x_{32}$: $u_3{+}v_2 = 3$. From $x_{31}$: $u_3{+}v_1 = 4 \Rightarrow u_3 = 1$. $v_2 = 3 - 1 = 2$. $u_2 = 1 - 2 = -1$.

Reduced costs (actual cells): $\delta_{12} = 5{-}0{-}2 = 3$, $\delta_{21} = 6{-}(-1){-}3 = 4$, $\delta_{23} = 4{-}(-1){-}2 = 3$, $\delta_{33} = 7{-}1{-}2 = 4$.

All $\delta_{ij} \geq 0$ for actual cells. $\Rightarrow$ the starting solution is already **optimal!**

Optimal cost (excluding dummy transports): **400**

**The problem was unbalanced; after adding $N_4$, the minimum-cost starting solution is already optimal with $K = 400$.**