Plain Vanilla Interest Rate Swap

VaR Calculation with Historical Time Series

Technical Documentation

English

Wolfgang Boehme

wolfgang.boehme@economic-modeling.de

www.economic-modeling.de

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Abstract

This document describes the mathematical foundations of Plain Vanilla Interest Rate Swap valuation and risk analysis using historical time series. The VaR calculation is based on the full covariance matrix of **zero rate changes**. The risk factors are the zero rates, not the discount factors or forward rates. The sensitivities are the discounted, time-weighted net cashflows: $\Delta_i = CF_{\text{net},i} \cdot t_i \cdot DF_i$. This methodology is identical to the bond calculator methodology; the only difference lies in the definition of cashflows (for swaps: net cashflows depend on the current yield curve).

1 Introduction

1.1 Clarification: Risk Factors

The risk factors are the zero rates, not the discount factors or forward rates!

For **all** fixed-income instruments (bonds, swaps, etc.):

$$PV = \sum_{i=1}^{n} CF_i \cdot DF(t_i) = \sum_{i=1}^{n} CF_i \cdot e^{-r_i \cdot t_i} \tag{1}$$

The sensitivity with respect to a zero rate r_i is:

$$\boxed{\frac{\partial PV}{\partial r_i} = -CF_i \cdot t_i \cdot DF_i} \tag{2}$$

Therefore, the **zero rates are the risk factors** and the **discounted, time-weighted cashflows are the sensitivities**.

1.2 Difference: Bond vs. Swap

Aspect	**Bond**	**Swap**
Risk factors	Zero rates	Zero rates
Covariance matrix	$\text{Cov}(\Delta r_i, \Delta r_j)$	$\text{Cov}(\Delta r_i, \Delta r_j)$
Sensitivities	$CF_i \cdot t_i \cdot DF_i$	$CF_{\text{net},i} \cdot t_i \cdot DF_i$
Cashflows	Fixed (coupon)	**Net CFs depend on yield curve**

Table 1: Bond vs. Swap – SAME risk factors!

The **only difference**:

- **Bond:** Cashflows independent of yield curve (fixed coupon)
- **Swap:** Net cashflows depend on current yield curve (floating leg)

1.3 Plain Vanilla Interest Rate Swap

A Plain Vanilla Interest Rate Swap is a derivative contract in which two parties exchange interest payments:

- **Fixed rate payer:** Pays a fixed interest rate
- **Floating rate payer:** Pays a variable interest rate (e.g. EURIBOR)

1.4 Characteristics

- **Notional amount:** N (EUR)
- **Fixed interest rate:** r_{fixed} (p.a.)
- **Maturity:** T (years)
- **Payment frequency:** f (1 = annual, 2 = semi-annual, 4 = quarterly)
- **Number of payments:** $n = \lfloor T \cdot f \rfloor$

1.5 Payment Structure

Payment dates: $t_1, t_2, \ldots, t_n$ with $t_i = i \cdot \frac{1}{f}$

Fixed leg (for fixed rate payer):

$$CF_{\text{fixed},i} = -N \cdot \frac{r_{\text{fixed}}}{f} \quad \text{for all } i \tag{3}$$

Floating leg (for fixed rate payer):

$$CF_{\text{floating},i} = +N \cdot \frac{FR_i}{f} \tag{4}$$

where FR_i is the forward rate for period i (from the current yield curve).

Net cashflow:

$$\boxed{CF_{\text{net},i} = CF_{\text{floating},i} + CF_{\text{fixed},i} = N \cdot \frac{FR_i - r_{\text{fixed}}}{f}} \tag{5}$$

The net cashflows depend on the current yield curve!

2 Discount Factors

2.1 Definition

The discount factor for maturity t is defined as:

$$DF(t) = e^{-r(t) \cdot t} \tag{6}$$

where $r(t)$ is the continuous zero rate (spot rate) for maturity t.

2.2 Yield Curve Interpolation

Given support points $(T_1, r_1), (T_2, r_2), \ldots, (T_m, r_m)$.

For a point in time t with $T_j \leq t \leq T_{j+1}$, we use linear interpolation:

$$r(t) = r_j + \frac{r_{j+1} - r_j}{T_{j+1} - T_j} \cdot (t - T_j) \tag{7}$$

Extrapolation:

- Before first tenor: $r(t) = r_1 \cdot \frac{t}{T_1}$ (proportional)
- After last tenor: $r(t) = r_m$ (constant)

3 Forward Rates

3.1 Definition

Forward rates are required to value the floating leg. They are implied interest rates derived from the discount factors.

3.2 Calculation from Discount Factors

For two points in time t_1 and t_2 with $t_1 < t_2$:

$$\boxed{FR(t_1, t_2) = \frac{DF(t_1) - DF(t_2)}{DF(t_2) \cdot (t_2 - t_1)}} \tag{8}$$

3.3 For Swap Payments

For the i-th payment at time t_i:

$$FR_i = FR(t_{i-1}, t_i) = \frac{DF(t_{i-1}) - DF(t_i)}{DF(t_i) \cdot (t_i - t_{i-1})} \tag{9}$$

Special case for the first payment ($t_0 = 0$, $DF(0) = 1$):

$$FR_1 = \frac{1 - DF(t_1)}{DF(t_1) \cdot t_1} \tag{10}$$

3.4 Role of Forward Rates

Forward rates are used solely to value the floating leg!

- Forward rates determine the net cashflows
- Net cashflows are the sensitivities
- **However:** The risk factors are the discount factors!

4 Swap Valuation

4.1 Present Value

The present value of a swap is:

$$\boxed{PV = \sum_{i=1}^{n} CF_{\text{net},i} \cdot DF(t_i)} \tag{11}$$

Expanded:

$$PV = \sum_{i=1}^{n} N \cdot \frac{FR_i - r_{\text{fixed}}}{f} \cdot DF(t_i) \tag{12}$$

4.2 Sensitivity

The sensitivity of PV with respect to a zero rate r_i is:

$$\boxed{\frac{\partial PV}{\partial r_i} = -CF_{\text{net},i} \cdot t_i \cdot DF_i} \tag{13}$$

where:

- $CF_{\text{net},i}$ = net cashflow at time t_i
- t_i = time in years (duration effect)
- $DF_i = e^{-r_i \cdot t_i}$ = discount factor

This is the fundamental equation for the VaR calculation!

5 VaR Methodology: Historical Time Series

5.1 Basic Principle

The covariance matrix is constructed from zero rate changes, not from discount factors or forward rates!

Why? Because the zero rates are the risk factors (see sensitivity).

5.2 Step 1: Discount Factor Time Series

Input: Historical interest rate time series (251 days)

Time series matrix: $r_{t,j}$ for day $t = 1, \ldots, 251$ and tenor $j = 1, \ldots, m$

For each day t and each payment date i:

1. Interpolate interest rate to payment date:

$$r(t, t_i) = \text{interpolate}(t_i, T_1, \ldots, T_m, r_{t,1}, \ldots, r_{t,m}) \tag{14}$$

2. Calculate discount factor:

$$DF(t, t_i) = e^{-r(t,t_i) \cdot t_i} \tag{15}$$

Result: Discount factor time series matrix $251 \times n$

$$\begin{pmatrix} DF(1,t_1) & DF(1,t_2) & \cdots & DF(1,t_n) \\ DF(2,t_1) & DF(2,t_2) & \cdots & DF(2,t_n) \\ \vdots & \vdots & \ddots & \vdots \\ DF(251,t_1) & DF(251,t_2) & \cdots & DF(251,t_n) \end{pmatrix} \tag{16}$$

5.3 Step 2: Log-Returns of Zero Rates

For each day $t = 2, \ldots, 251$ and each payment date $i = 1, \ldots, n$:

$$\boxed{\text{Return}(t, i) = r(t, t_i) - r(t-1, t_i)} \tag{17}$$

where $r(t, t_i)$ is the interpolated zero rate for maturity t_i on day t.
Alternative (log-returns):

$$\text{Return}(t, i) = \ln\left[\frac{r(t, t_i)}{r(t-1, t_i)}\right] \tag{18}$$

Result: Returns matrix $250 \times n$

5.4 Step 3: Mean Values

For each payment date i:

$$\mu_i = \frac{1}{250} \sum_{t=2}^{251} \text{Return}(t, i) \tag{19}$$

5.5 Step 4: Covariance Matrix of Zero Rates

The covariance between zero rate r_i and r_j:

$$\boxed{\Sigma_{ij} = \frac{1}{249} \sum_{t=2}^{251} \left[\text{Return}(t, i) - \mu_i\right] \cdot \left[\text{Return}(t, j) - \mu_j\right]} \tag{20}$$

Result: Covariance matrix Σ of dimension $n \times n$
This is the covariance matrix of zero rate changes!

5.6 Step 5: Sensitivity Vector

The sensitivity vector consists of the **discounted, time-weighted** net cashflows:

$$\mathbf{w} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{pmatrix} = \begin{pmatrix} CF_{\text{net},1} \cdot t_1 \cdot DF_1 \\ CF_{\text{net},2} \cdot t_2 \cdot DF_2 \\ \vdots \\ CF_{\text{net},n} \cdot t_n \cdot DF_n \end{pmatrix} \tag{21}$$

where:

- $CF_{\text{net},i}$ = net cashflow at time t_i
- t_i = time in years
- $DF_i = e^{-r_i \cdot t_i}$ = discount factor
- $\Delta_i = -\frac{\partial PV}{\partial r_i}$ = sensitivity (defined positive)

where each $CF_{\text{net},i}$ is calculated from the current yield curve:

$$CF_{\text{net},i} = N \cdot \frac{FR_i - r_{\text{fixed}}}{f} \tag{22}$$

5.7 Step 6: Portfolio Variance

$$\boxed{\sigma_P^2 = \mathbf{w}^T \Sigma \mathbf{w} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot \Sigma_{ij} \cdot w_j} \tag{23}$$

where:

- $\mathbf{w}$ = sensitivities ($\Delta_i = CF_{\text{net},i} \cdot t_i \cdot DF_i$)
- Σ = covariance matrix of zero rate changes

5.8 Step 7: Value at Risk

For confidence level α and time horizon h days:

$$\boxed{VaR_\alpha = z_\alpha \cdot \sqrt{\sigma_P^2} \cdot \sqrt{h}} \tag{24}$$

where z_α is the α-quantile of the standard normal distribution.

6 Complete Worked Example

6.1 Parameters

Swap specification:

- Notional amount: $N = 1{,}000{,}000$ EUR
- Fixed interest rate: $r_{\text{fixed}} = 3.5\%$ p.a.
- Maturity: $T = 5$ years
- Payment frequency: $f = 1$ (annual)
- Number of payments: $n = 5$

Current yield curve (last row of time series):

Tenor	Interest rate (%)
1Y	2.88
2Y	2.94
3Y	3.04
4Y	3.32
5Y	3.88

Time series: 251 days with 5 tenors (1Y–5Y)

6.2 Step 1: Discount Factors (today)

From the current yield curve:

$$DF(1) = e^{-0.0288 \cdot 1} = 0.971609 \tag{25}$$
$$DF(2) = e^{-0.0294 \cdot 2} = 0.943037 \tag{26}$$
$$DF(3) = e^{-0.0304 \cdot 3} = 0.912702 \tag{27}$$
$$DF(4) = e^{-0.0332 \cdot 4} = 0.875633 \tag{28}$$
$$DF(5) = e^{-0.0388 \cdot 5} = 0.823741 \tag{29}$$

6.3 Step 2: Forward Rates (today)

$$FR_1 = \frac{1 - 0.971609}{0.971609 \cdot 1} = 0.029220 = 2.922\% \tag{30}$$
$$FR_2 = \frac{0.971609 - 0.943037}{0.943037 \cdot 1} = 0.030304 = 3.030\% \tag{31}$$
$$FR_3 = \frac{0.943037 - 0.912702}{0.912702 \cdot 1} = 0.033241 = 3.324\% \tag{32}$$
$$FR_4 = \frac{0.912702 - 0.875633}{0.875633 \cdot 1} = 0.042336 = 4.234\% \tag{33}$$
$$FR_5 = \frac{0.875633 - 0.823741}{0.823741 \cdot 1} = 0.063009 = 6.301\% \tag{34}$$

6.4 Step 3: Net Cashflows (today)

Fixed cashflows:

$$CF_{\text{fixed},i} = -1{,}000{,}000 \cdot \frac{0.035}{1} = -35{,}000 \text{ EUR} \tag{35}$$

Floating cashflows:

$$CF_{\text{floating},1} = 1{,}000{,}000 \cdot \frac{0.029220}{1} = 29{,}220 \text{ EUR} \tag{36}$$
$$CF_{\text{floating},2} = 1{,}000{,}000 \cdot \frac{0.030304}{1} = 30{,}304 \text{ EUR} \tag{37}$$
$$CF_{\text{floating},3} = 1{,}000{,}000 \cdot \frac{0.033241}{1} = 33{,}241 \text{ EUR} \tag{38}$$
$$CF_{\text{floating},4} = 1{,}000{,}000 \cdot \frac{0.042336}{1} = 42{,}336 \text{ EUR} \tag{39}$$
$$CF_{\text{floating},5} = 1{,}000{,}000 \cdot \frac{0.063009}{1} = 63{,}009 \text{ EUR} \tag{40}$$

Net cashflows (sensitivity vector):

$$\mathbf{w} = \begin{pmatrix} 29{,}220 - 35{,}000 \\ 30{,}304 - 35{,}000 \\ 33{,}241 - 35{,}000 \\ 42{,}336 - 35{,}000 \\ 63{,}009 - 35{,}000 \end{pmatrix} = \begin{pmatrix} -5{,}780 \\ -4{,}696 \\ -1{,}759 \\ +7{,}336 \\ +28{,}009 \end{pmatrix} \text{ EUR} \tag{41}$$

6.5 Step 4: Present Value (today)

$$PV = \sum_{i=1}^{5} w_i \cdot DF(t_i) \tag{42}$$

$$= (-5{,}780) \cdot 0.971609 + (-4{,}696) \cdot 0.943037 \tag{43}$$

$$+ (-1{,}759) \cdot 0.912702 + 7{,}336 \cdot 0.875633 \tag{44}$$

$$+ 28{,}009 \cdot 0.823741 \tag{45}$$

$$= -5{,}616 - 4{,}428 - 1{,}606 + 6{,}424 + 23{,}069 \tag{46}$$

$$= \boxed{17{,}843 \text{ EUR}} \tag{47}$$

6.6 Step 5: Discount Factor Time Series

Example: First 5 days of time series

Day	1Y (%)	2Y (%)	3Y (%)	4Y (%)	5Y (%)
1	2.50	2.70	2.85	2.95	3.00
2	2.54	2.74	2.91	3.07	3.15
3	2.48	2.68	2.83	2.93	2.98
4	2.52	2.72	2.88	3.00	3.10
5	2.56	2.76	2.94	3.10	3.20
⋮	⋮	⋮	⋮	⋮	⋮
251	2.88	2.94	3.04	3.32	3.88

Table 2: Time series: Interest rates (first 5 days, example)

Calculation of discount factors:
For day 1, payment date 1 year:

$$DF(1,1) = e^{-0.0250 \cdot 1} = 0.975310 \tag{48}$$

For day 1, payment date 2 years:

$$DF(1,2) = e^{-0.0270 \cdot 2} = 0.947491 \tag{49}$$

Analogously for all 251 days and all 5 payment dates.
Result: DF matrix 251×5

6.7 Step 6: Log-Returns of Discount Factors

For day 2:

Day	$DF(t_1)$	$DF(t_2)$	$DF(t_3)$	$DF(t_4)$	$DF(t_5)$
1	0.975310	0.947491	0.917626	0.886374	0.860708
2	0.974929	0.946735	0.915941	0.882489	0.854075
3	0.975505	0.947867	0.918055	0.887805	0.862260
4	0.975119	0.947113	0.916512	0.882603	0.856531
5	0.974738	0.946367	0.915011	0.879165	0.851138
⋮	⋮	⋮	⋮	⋮	⋮
251	0.971609	0.943037	0.912702	0.875633	0.823741

Table 3: Discount factor time series (first 5 days, example)

$$\text{Return}(2,1) = \ln\left[\frac{0.974929}{0.975310}\right] = -0.000391 \tag{50}$$

$$\text{Return}(2,2) = \ln\left[\frac{0.946735}{0.947491}\right] = -0.000799 \tag{51}$$

$$\text{Return}(2,3) = \ln\left[\frac{0.915941}{0.917626}\right] = -0.001838 \tag{52}$$

$$\text{Return}(2,4) = \ln\left[\frac{0.882489}{0.886374}\right] = -0.004401 \tag{53}$$

$$\text{Return}(2,5) = \ln\left[\frac{0.854075}{0.860708}\right] = -0.007770 \tag{54}$$

Analogously for days 3 to 251.
Result: Returns matrix 250×5

6.8 Step 7: Mean Values of Returns

$$\mu_1 = \frac{1}{250}\sum_{t=2}^{251}\text{Return}(t,1) \approx 0.0002 \tag{55}$$

$$\mu_2 = \frac{1}{250}\sum_{t=2}^{251}\text{Return}(t,2) \approx 0.0003 \tag{56}$$

$$\mu_3 = \frac{1}{250}\sum_{t=2}^{251}\text{Return}(t,3) \approx 0.0005 \tag{57}$$

$$\mu_4 = \frac{1}{250}\sum_{t=2}^{251}\text{Return}(t,4) \approx 0.0008 \tag{58}$$

$$\mu_5 = \frac{1}{250}\sum_{t=2}^{251}\text{Return}(t,5) \approx 0.0012 \tag{59}$$

6.9 Step 8: Covariance Matrix of Zero Rate Changes

Calculation:

For each pair (i, j):

$$\Sigma_{ij} = \frac{1}{249} \sum_{t=2}^{251} [\text{Return}(t, i) - \mu_i] \cdot [\text{Return}(t, j) - \mu_j] \tag{60}$$

Example: Σ_{11} (variance of DF_1)

$$\Sigma_{11} = \frac{1}{249} \sum_{t=2}^{251} [\text{Return}(t, 1) - 0.0002]^2 \tag{61}$$

$$\approx 0.000045 \tag{62}$$

Example: Σ_{12} (covariance of DF_1 and DF_2)

$$\Sigma_{12} = \frac{1}{249} \sum_{t=2}^{251} [\text{Return}(t, 1) - 0.0002] \cdot [\text{Return}(t, 2) - 0.0003] \tag{63}$$

$$\approx 0.000085 \tag{64}$$

Full covariance matrix (hypothetical example):

$$\Sigma = \begin{pmatrix} 0.000045 & 0.000085 & 0.000120 & 0.000145 & 0.000160 \\ 0.000085 & 0.000180 & 0.000240 & 0.000290 & 0.000320 \\ 0.000120 & 0.000240 & 0.000360 & 0.000435 & 0.000480 \\ 0.000145 & 0.000290 & 0.000435 & 0.000580 & 0.000640 \\ 0.000160 & 0.000320 & 0.000480 & 0.000640 & 0.000800 \end{pmatrix} \tag{65}$$

Properties:

- Symmetric: $\Sigma_{ij} = \Sigma_{ji}$
- Diagonal increases: $\Sigma_{11} < \Sigma_{22} < \cdots < \Sigma_{55}$
- Typical for discount factors: longer maturities have higher variance

6.10 Step 9: Correlation Matrix

Standard deviations:

$$\sigma_1 = \sqrt{0.000045} = 0.00671 = 0.671\% \tag{66}$$

$$\sigma_2 = \sqrt{0.000180} = 0.01342 = 1.342\% \tag{67}$$

$$\sigma_3 = \sqrt{0.000360} = 0.01897 = 1.897\% \tag{68}$$

$$\sigma_4 = \sqrt{0.000580} = 0.02408 = 2.408\% \tag{69}$$

$$\sigma_5 = \sqrt{0.000800} = 0.02828 = 2.828\% \tag{70}$$

Correlations:

$$\rho_{ij} = \frac{\Sigma_{ij}}{\sigma_i \cdot \sigma_j} \tag{71}$$

Example: ρ_{12}

$$\rho_{12} = \frac{0.000085}{0.00671 \cdot 0.01342} = 0.944 \tag{72}$$

Full correlation matrix:

$$\mathbf{R} = \begin{pmatrix} 1.000 & 0.944 & 0.943 & 0.898 & 0.843 \\ 0.944 & 1.000 & 0.997 & 0.951 & 0.893 \\ 0.943 & 0.997 & 1.000 & 0.995 & 0.946 \\ 0.898 & 0.951 & 0.995 & 1.000 & 0.983 \\ 0.843 & 0.893 & 0.946 & 0.983 & 1.000 \end{pmatrix} \tag{73}$$

Observations:

- Very high correlation between adjacent discount factors ($\rho > 0.94$)
- Correlation decreases with distance (but remains high)
- Typical for the yield curve structure

6.11 Step 10: Portfolio Variance

Matrix multiplication: $\sigma_P^2 = \mathbf{w}^T \Sigma \mathbf{w}$

Detailed calculation:

First row of $\mathbf{w}^T\Sigma$:

$$\begin{aligned} [\mathbf{w}^T\Sigma]_1 &= w_1\Sigma_{11} + w_2\Sigma_{21} + w_3\Sigma_{31} + w_4\Sigma_{41} + w_5\Sigma_{51} && (74)\\ &= (-5{,}780) \cdot 0.000045 + (-4{,}696) \cdot 0.000085 && (75)\\ &\quad + (-1{,}759) \cdot 0.000120 + 7{,}336 \cdot 0.000145 && (76)\\ &\quad + 28{,}009 \cdot 0.000160 && (77)\\ &= -0.260 - 0.399 - 0.211 + 1.064 + 4.481 && (78)\\ &= 4.675 && (79) \end{aligned}$$

Analogously for rows 2–5:

$$\begin{aligned} [\mathbf{w}^T\Sigma]_2 &= 8.805 && (80)\\ [\mathbf{w}^T\Sigma]_3 &= 12.450 && (81)\\ [\mathbf{w}^T\Sigma]_4 &= 15.038 && (82)\\ [\mathbf{w}^T\Sigma]_5 &= 16.560 && (83) \end{aligned}$$

Dot product with w:

$$\begin{aligned} \sigma_P^2 &= [\mathbf{w}^T\Sigma] \cdot \mathbf{w} && (84)\\ &= 4.675 \cdot (-5{,}780) + 8.805 \cdot (-4{,}696) && (85)\\ &\quad + 12.450 \cdot (-1{,}759) + 15.038 \cdot 7{,}336 && (86)\\ &\quad + 16.560 \cdot 28{,}009 && (87)\\ &= -27.02 - 41.35 - 21.90 + 110.30 + 463.83 && (88)\\ &= \boxed{483.86 \text{ EUR}^2} && (89) \end{aligned}$$

Portfolio standard deviation:

$$\sigma_P = \sqrt{483.86} = \boxed{22.00 \text{ EUR}} \tag{90}$$

6.12 Step 11: Value at Risk

For $\alpha = 0.99$ (99%) and $h = 1$ day:

$$z_{0.99} = 2.326 \tag{91}$$

$$\boxed{VaR_{0.99} = 2.326 \cdot 22.00 \cdot \sqrt{1} = 51.17 \text{ EUR}} \tag{92}$$

Interpretation: With 99% probability, the 1-day loss will not exceed 51.17 EUR.

6.13 Summary of Results

Key figure	**Value**
Present value	17,843 EUR
Sensitivities ($\Delta_i = CF_{\mathbf{net},i} \cdot t_i \cdot DF_i$):	
w_1	-5,780 EUR
w_2	-4,696 EUR
w_3	-1,759 EUR
w_4	+7,336 EUR
w_5	+28,009 EUR
Covariance matrix (variances):	
Σ_{11}	0.000045
Σ_{22}	0.000180
Σ_{33}	0.000360
Σ_{44}	0.000580
Σ_{55}	0.000800
Portfolio variance	483.86 EUR^2
Portfolio σ	22.00 EUR
VaR (99%, 1 day)	51.17 EUR

Table 4: Complete results

7 Detailed Example: Matrix Construction

7.1 Step-by-Step Guide

Given: Time series with 251 days, 5 tenors (1Y–5Y)
Goal: Covariance matrix Σ of dimension 5×5

7.2 Step 1: Build the DF Matrix

For each day t and each tenor j:

$$DF(t, T_j) = e^{-r_{t,j} \cdot T_j} \tag{93}$$

Example day 1:

$$DF(1,1) = e^{-0.0250 \cdot 1} = 0.975310 \tag{94}$$
$$DF(1,2) = e^{-0.0270 \cdot 2} = 0.947491 \tag{95}$$
$$DF(1,3) = e^{-0.0285 \cdot 3} = 0.917626 \tag{96}$$
$$DF(1,4) = e^{-0.0295 \cdot 4} = 0.886374 \tag{97}$$
$$DF(1,5) = e^{-0.0300 \cdot 5} = 0.860708 \tag{98}$$

Example day 2:

$$DF(2,1) = e^{-0.0254 \cdot 1} = 0.974929 \tag{99}$$
$$DF(2,2) = e^{-0.0274 \cdot 2} = 0.946735 \tag{100}$$
$$DF(2,3) = e^{-0.0291 \cdot 3} = 0.915941 \tag{101}$$
$$DF(2,4) = e^{-0.0307 \cdot 4} = 0.882489 \tag{102}$$
$$DF(2,5) = e^{-0.0315 \cdot 5} = 0.854075 \tag{103}$$

Analogously for all 251 days $\rightarrow$ DF matrix 251×5

7.3 Step 2: Calculate Returns

For each day $t \geq 2$ and each tenor j:

$$\text{Return}(t,j) = \ln\left[\frac{DF(t,T_j)}{DF(t-1,T_j)}\right] \tag{104}$$

Day 2, all tenors:

$$\text{Return}(2,1) = \ln\left[\frac{0.974929}{0.975310}\right] = -0.000391 \tag{105}$$
$$\text{Return}(2,2) = \ln\left[\frac{0.946735}{0.947491}\right] = -0.000799 \tag{106}$$
$$\text{Return}(2,3) = \ln\left[\frac{0.915941}{0.917626}\right] = -0.001838 \tag{107}$$
$$\text{Return}(2,4) = \ln\left[\frac{0.882489}{0.886374}\right] = -0.004401 \tag{108}$$
$$\text{Return}(2,5) = \ln\left[\frac{0.854075}{0.860708}\right] = -0.007770 \tag{109}$$

Day 3, all tenors:

$$\text{Return}(3,1) = \ln\left[\frac{0.975505}{0.974929}\right] = +0.000591 \tag{110}$$
$$\text{Return}(3,2) = \ln\left[\frac{0.947867}{0.946735}\right] = +0.001196 \tag{111}$$
$$\text{Return}(3,3) = \ln\left[\frac{0.918055}{0.915941}\right] = +0.002309 \tag{112}$$
$$\text{Return}(3,4) = \ln\left[\frac{0.887805}{0.882489}\right] = +0.006036 \tag{113}$$
$$\text{Return}(3,5) = \ln\left[\frac{0.862260}{0.854075}\right] = +0.009602 \tag{114}$$

Analogously for days 4 to 251 $\rightarrow$ returns matrix 250×5

7.4 Step 3: Mean Values

$$\mu_j = \frac{1}{250} \sum_{t=2}^{251} \text{Return}(t, j) \tag{115}$$

Example μ_1:

$$\mu_1 = \frac{1}{250}[(-0.000391) + 0.000591 + \text{Return}(4,1) + \cdots + \text{Return}(251,1)] \tag{116}$$

$$\approx 0.0002 \tag{117}$$

Analogously $\mu_2, \ldots, \mu_5$

7.5 Step 4: Covariance Matrix Element by Element

$$\Sigma_{ij} = \frac{1}{249} \sum_{t=2}^{251} [\text{Return}(t, i) - \mu_i] \cdot [\text{Return}(t, j) - \mu_j] \tag{118}$$

Example Σ_{11} (variance):

$$\Sigma_{11} = \frac{1}{249} \Big\{ [(-0.000391) - 0.0002]^2 + [0.000591 - 0.0002]^2 \tag{119}$$

$$+ [\text{Return}(4,1) - 0.0002]^2 + \cdots + [\text{Return}(251,1) - 0.0002]^2 \Big\} \tag{120}$$

$$\approx 0.000045 \tag{121}$$

Example Σ_{12} (covariance):

$$\Sigma_{12} = \frac{1}{249} \Big\{ [(-0.000391) - 0.0002] \cdot [(-0.000799) - 0.0003] \tag{122}$$

$$+ [0.000591 - 0.0002] \cdot [0.001196 - 0.0003] \tag{123}$$

$$+ [\text{Return}(4,1) - 0.0002] \cdot [\text{Return}(4,2) - 0.0003] \tag{124}$$

$$+ \cdots \tag{125}$$

$$+ [\text{Return}(251,1) - 0.0002] \cdot [\text{Return}(251,2) - 0.0003] \Big\} \tag{126}$$

$$\approx 0.000085 \tag{127}$$

Example Σ_{13} (covariance):
Analogous to Σ_{12}, but with returns for tenor 3:

$$\Sigma_{13} = \frac{1}{249} \sum_{t=2}^{251} [\text{Return}(t, 1) - \mu_1] \cdot [\text{Return}(t, 3) - \mu_3] \approx 0.000120 \tag{128}$$

Analogously for all 25 elements of the matrix.

7.6 Step 5: Full Matrix

After calculating all elements:

$$\Sigma = \begin{pmatrix} 0.000045 & 0.000085 & 0.000120 & 0.000145 & 0.000160 \\ 0.000085 & 0.000180 & 0.000240 & 0.000290 & 0.000320 \\ 0.000120 & 0.000240 & 0.000360 & 0.000435 & 0.000480 \\ 0.000145 & 0.000290 & 0.000435 & 0.000580 & 0.000640 \\ 0.000160 & 0.000320 & 0.000480 & 0.000640 & 0.000800 \end{pmatrix} \tag{129}$$

This matrix contains the covariance of the discount factor returns!

8 Risk Metrics

8.1 DV01

$$DV01 = -[PV(r + 0.0001) - PV(r)] \tag{130}$$

For our example: $DV01 \approx 453$ EUR

8.2 Modified Duration

$$D_{\text{mod}} = \frac{DV01 \cdot 10{,}000}{|PV|} \tag{131}$$

For our example: $D_{\text{mod}} = \frac{453 \cdot 10{,}000}{17{,}843} \approx 253.8$ years

9 Comparison with Simplified Methods

9.1 Method Without Covariance Matrix

Simplified VaR calculation (e.g. in v11):

$$VaR \approx z_\alpha \cdot \sigma \cdot DV01 \cdot 10{,}000 \tag{132}$$

where σ is an estimated or historical average volatility.
Problems:

- Ignores covariance between payment dates
- Uses uniform volatility for all maturities
- Typically too low

9.2 Method with Full Covariance Matrix (v13)

$$VaR = z_\alpha \cdot \sqrt{\mathbf{w}^T \Sigma \mathbf{w}} \cdot \sqrt{h} \tag{133}$$

Advantages:

- Accounts for full covariance structure
- Different volatilities per maturity

- More realistic correlations
- Typically 20–50% higher than simplified methods

10 Semi-Annual Payments

10.1 Example

Parameters:

- 2-year maturity
- Semi-annual $\rightarrow$ 4 payments at $t = 0.5, 1.0, 1.5, 2.0$

Time series: Annual tenors (1Y, 2Y)
Problem: Need DFs for $t = 0.5$ and 1.5

10.2 Interpolation

For each day in the time series:
$r(t, 0.5)$**:** Extrapolation before 1Y

$$r(t, 0.5) = r(t, 1) \cdot \frac{0.5}{1.0} \tag{134}$$

$r(t, 1.5)$**:** Interpolation between 1Y and 2Y

$$r(t, 1.5) = r(t, 1) + \frac{r(t, 2) - r(t, 1)}{2 - 1} \cdot (1.5 - 1) \tag{135}$$

Then calculate DF time series for all 4 payment dates and build covariance matrix 4×4.

10.3 Covariance Matrix Calculation (Semi-Annual)

Full example: 2 years, semi-annual, 1M EUR, 3.5% fixed
Payment dates: $t_1 = 0.5$, $t_2 = 1.0$, $t_3 = 1.5$, $t_4 = 2.0$
Time series: 251 days with 2 tenors (1Y, 2Y)

10.3.1 Step 1: Interpolation for All Days

Day 1: Interest rates 2.50% (1Y), 2.70% (2Y)
Interpolation to payment dates:

$$r(1, 0.5) = 2.50\% \cdot \frac{0.5}{1.0} = 1.25\% \tag{136}$$

$$r(1, 1.0) = 2.50\% \quad \text{(support point)} \tag{137}$$

$$r(1, 1.5) = 2.50 + \frac{2.70 - 2.50}{2 - 1} \cdot (1.5 - 1) = 2.60\% \tag{138}$$

$$r(1, 2.0) = 2.70\% \quad \text{(support point)} \tag{139}$$

Day 2: Interest rates 2.54% (1Y), 2.74% (2Y)

$$r(2, 0.5) = 2.54\% \cdot 0.5 = 1.27\% \tag{140}$$
$$r(2, 1.0) = 2.54\% \tag{141}$$
$$r(2, 1.5) = 2.54 + \frac{2.74 - 2.54}{1} \cdot 0.5 = 2.64\% \tag{142}$$
$$r(2, 2.0) = 2.74\% \tag{143}$$

Day 3: Interest rates 2.48% (1Y), 2.68% (2Y)

$$r(3, 0.5) = 2.48\% \cdot 0.5 = 1.24\% \tag{144}$$
$$r(3, 1.0) = 2.48\% \tag{145}$$
$$r(3, 1.5) = 2.48 + \frac{2.68 - 2.48}{1} \cdot 0.5 = 2.58\% \tag{146}$$
$$r(3, 2.0) = 2.68\% \tag{147}$$

Analogously for all 251 days.

10.3.2 Step 2: Discount Factors for All Days

Day 1:

$$DF(1, 0.5) = e^{-0.0125 \cdot 0.5} = 0.993769 \tag{148}$$
$$DF(1, 1.0) = e^{-0.0250 \cdot 1.0} = 0.975310 \tag{149}$$
$$DF(1, 1.5) = e^{-0.0260 \cdot 1.5} = 0.961716 \tag{150}$$
$$DF(1, 2.0) = e^{-0.0270 \cdot 2.0} = 0.947491 \tag{151}$$

Day 2:

$$DF(2, 0.5) = e^{-0.0127 \cdot 0.5} = 0.993681 \tag{152}$$
$$DF(2, 1.0) = e^{-0.0254 \cdot 1.0} = 0.974929 \tag{153}$$
$$DF(2, 1.5) = e^{-0.0264 \cdot 1.5} = 0.961135 \tag{154}$$
$$DF(2, 2.0) = e^{-0.0274 \cdot 2.0} = 0.946735 \tag{155}$$

Day 3:

$$DF(3, 0.5) = e^{-0.0124 \cdot 0.5} = 0.993808 \tag{156}$$
$$DF(3, 1.0) = e^{-0.0248 \cdot 1.0} = 0.975505 \tag{157}$$
$$DF(3, 1.5) = e^{-0.0258 \cdot 1.5} = 0.962004 \tag{158}$$
$$DF(3, 2.0) = e^{-0.0268 \cdot 2.0} = 0.947867 \tag{159}$$

Result after all 251 days: DF matrix 251×4

Day	$DF(t, 0.5)$	$DF(t, 1.0)$	$DF(t, 1.5)$	$DF(t, 2.0)$
1	0.993769	0.975310	0.961716	0.947491
2	0.993681	0.974929	0.961135	0.946735
3	0.993808	0.975505	0.962004	0.947867
4	0.993720	0.975119	0.961425	0.947113
5	0.993633	0.974738	0.960849	0.946367
⋮	⋮	⋮	⋮	⋮
251	0.992803	0.971609	0.957211	0.943037

Table 5: Discount factor time series (semi-annual, first 5 days)

10.3.3 Step 3: Returns of Discount Factors

Day 2:

$$\text{Return}(2,1) = \ln\left[\frac{DF(2,0.5)}{DF(1,0.5)}\right] = \ln\left[\frac{0.993681}{0.993769}\right] = -0.000089 \tag{160}$$

$$\text{Return}(2,2) = \ln\left[\frac{DF(2,1.0)}{DF(1,1.0)}\right] = \ln\left[\frac{0.974929}{0.975310}\right] = -0.000391 \tag{161}$$

$$\text{Return}(2,3) = \ln\left[\frac{DF(2,1.5)}{DF(1,1.5)}\right] = \ln\left[\frac{0.961135}{0.961716}\right] = -0.000604 \tag{162}$$

$$\text{Return}(2,4) = \ln\left[\frac{DF(2,2.0)}{DF(1,2.0)}\right] = \ln\left[\frac{0.946735}{0.947491}\right] = -0.000799 \tag{163}$$

Analogously for days 3 to 251 $\rightarrow$ returns matrix 250×4

10.3.4 Step 4: Covariance Matrix 4×4

$$\Sigma = \begin{pmatrix} 0.000012 & 0.000023 & 0.000033 & 0.000042 \\ 0.000023 & 0.000045 & 0.000065 & 0.000083 \\ 0.000033 & 0.000065 & 0.000098 & 0.000125 \\ 0.000042 & 0.000083 & 0.000125 & 0.000180 \end{pmatrix} \tag{164}$$

10.3.5 Step 5: Correlation Matrix 4×4

$$\mathbf{R} = \begin{pmatrix} 1.000 & 0.991 & 0.963 & 0.905 \\ 0.991 & 1.000 & 0.978 & 0.944 \\ 0.963 & 0.978 & 1.000 & 0.978 \\ 0.905 & 0.944 & 0.978 & 1.000 \end{pmatrix} \tag{165}$$

Observations:

- Very high correlation (all > 0.9)
- Adjacent dates have the highest correlation ($\rho_{12} = 0.991$, $\rho_{23} = 0.978$, $\rho_{34} = 0.978$)
- Correlation decreases with distance ($\rho_{14} = 0.905$)
- Typical for the interest rate term structure

10.3.6 Step 7: Net Cashflows (today)

From current yield curve (last row: 2.88%, 2.94%):

Forward rates:

$$FR_1 = \frac{1 - DF(0.5)}{DF(0.5) \cdot 0.5} = \frac{1 - 0.992803}{0.992803 \cdot 0.5} = 0.014501 = 1.450\% \tag{166}$$

$$FR_2 = \frac{DF(0.5) - DF(1.0)}{DF(1.0) \cdot 0.5} = \frac{0.992803 - 0.971609}{0.971609 \cdot 0.5} = 0.043647 = 4.365\% \tag{167}$$

$$FR_3 = \frac{DF(1.0) - DF(1.5)}{DF(1.5) \cdot 0.5} = \frac{0.971609 - 0.957211}{0.957211 \cdot 0.5} = 0.030108 = 3.011\% \tag{168}$$

$$FR_4 = \frac{DF(1.5) - DF(2.0)}{DF(2.0) \cdot 0.5} = \frac{0.957211 - 0.943037}{0.943037 \cdot 0.5} = 0.030057 = 3.006\% \tag{169}$$

Cashflows (semi-annual):

Fixed cashflows: $CF_{\text{fixed},i} = -1{,}000{,}000 \cdot \frac{0.035}{2} = -17{,}500$ EUR

Floating cashflows:

$$CF_{\text{floating},1} = 1{,}000{,}000 \cdot \frac{0.014501}{2} = 7{,}251 \text{ EUR} \tag{170}$$

$$CF_{\text{floating},2} = 1{,}000{,}000 \cdot \frac{0.043647}{2} = 21{,}824 \text{ EUR} \tag{171}$$

$$CF_{\text{floating},3} = 1{,}000{,}000 \cdot \frac{0.030108}{2} = 15{,}054 \text{ EUR} \tag{172}$$

$$CF_{\text{floating},4} = 1{,}000{,}000 \cdot \frac{0.030057}{2} = 15{,}029 \text{ EUR} \tag{173}$$

Net cashflows (sensitivity vector):

$$\mathbf{w} = \begin{pmatrix} 7{,}251 - 17{,}500 \\ 21{,}824 - 17{,}500 \\ 15{,}054 - 17{,}500 \\ 15{,}029 - 17{,}500 \end{pmatrix} = \begin{pmatrix} -10{,}249 \\ +4{,}324 \\ -2{,}446 \\ -2{,}471 \end{pmatrix} \text{ EUR} \tag{174}$$

10.3.7 Step 8: Portfolio Variance

Matrix multiplication: $\sigma_P^2 = \mathbf{w}^T \Sigma \mathbf{w}$

Row 1 of $\mathbf{w}^T\Sigma$:

$$[\mathbf{w}^T\Sigma]_1 = w_1\Sigma_{11} + w_2\Sigma_{21} + w_3\Sigma_{31} + w_4\Sigma_{41} \tag{175}$$

$$= (-10{,}249) \cdot 0.000012 + 4{,}324 \cdot 0.000023 \tag{176}$$

$$+ (-2{,}446) \cdot 0.000033 + (-2{,}471) \cdot 0.000042 \tag{177}$$

$$= -0.123 + 0.099 - 0.081 - 0.104 \tag{178}$$

$$= -0.209 \tag{179}$$

Analogously:

$$[\mathbf{w}^T\Sigma]_2 = -0.347 \tag{180}$$

$$[\mathbf{w}^T\Sigma]_3 = -0.449 \tag{181}$$

$$[\mathbf{w}^T\Sigma]_4 = -0.589 \tag{182}$$

Dot product:

$$\sigma_P^2 = [\mathbf{w}^T \Sigma] \cdot \mathbf{w} \tag{183}$$
$$= (-0.209) \cdot (-10{,}249) + (-0.347) \cdot 4{,}324 \tag{184}$$
$$+ (-0.449) \cdot (-2{,}446) + (-0.589) \cdot (-2{,}471) \tag{185}$$
$$= 2.142 - 1.500 + 1.098 + 1.455 \tag{186}$$
$$= \boxed{3.195 \text{ EUR}^2} \tag{187}$$

Portfolio standard deviation:

$$\sigma_P = \sqrt{3.195} = \boxed{1.79 \text{ EUR}} \tag{188}$$

10.3.8 Step 9: Value at Risk

For $\alpha = 0.99$ (99%) and $h = 1$ day:

$$z_{0.99} = 2.326 \tag{189}$$

$$\boxed{VaR_{0.99} = 2.326 \cdot 1.79 \cdot \sqrt{1} = 4.16 \text{ EUR}} \tag{190}$$

Interpretation: With 99% probability, the 1-day loss will not exceed 4.16 EUR.

10.3.9 Summary: Semi-Annual Example

Parameter	**Value**
Notional amount	1,000,000 EUR
Fixed interest rate	3.5% p.a.
Maturity	2 years
Frequency	Semi-annual (4 payments)
Sensitivities ($\Delta_i = CF_{\mathbf{net},i} \cdot t_i \cdot DF_i$):	
w_1 (at $t = 0.5$)	-10,249 EUR
w_2 (at $t = 1.0$)	+4,324 EUR
w_3 (at $t = 1.5$)	-2,446 EUR
w_4 (at $t = 2.0$)	-2,471 EUR
Covariance matrix (variances):	
Σ_{11}	0.000012
Σ_{22}	0.000045
Σ_{33}	0.000098
Σ_{44}	0.000180
Portfolio variance	3.195 EUR^2
Portfolio σ	1.79 EUR
VaR (99%, 1 day)	4.16 EUR

Table 6: Results: Semi-annual example (2 years)

Key points:

1. **Interpolation:** Interest rates for $t = 0.5$ and 1.5 from 1Y and 2Y tenors
2. **DF matrix:** 251×4 (251 days, 4 payment dates)
3. **Covariance matrix:** 4×4 (from DF returns)
4. **Variances:** Increase with maturity ($\Sigma_{11} < \Sigma_{44}$)
5. **Correlations:** Very high (> 0.9), typical for the term structure

11 Implementation Notes

11.1 Time Series Format

CSV structure:

```
Date,1Y,2Y,3Y,4Y,5Y
2023-12-18,2.50,2.70,2.85,2.95,3.00
2023-12-19,2.54,2.74,2.91,3.07,3.15
...
2024-12-02,2.88,2.94,3.04,3.32,3.88
```

Requirements:

- At least 251 data rows (250 returns)
- Header with tenors
- Interest rates in percent
- Last row = current yield curve

11.2 Numerical Stability

- Covariance matrix: check for positive semi-definiteness
- If problems arise: regularisation (small diagonal addition)
- Use robust algorithms

12 Validation

12.1 Plausibility Checks

Discount factors:

- $0 < DF(t) < 1$ for all $t > 0$
- $DF(t_1) > DF(t_2)$ for $t_1 < t_2$

Returns:

- Mean close to 0

- No extreme outliers

Covariance matrix:

- Symmetric
- Positive semi-definite
- Diagonal increases (typical): $\Sigma_{11} < \Sigma_{22} < \cdots < \Sigma_{nn}$

VaR:

- $VaR > 0$
- Increases with α and h
- Increases linearly with notional amount

13 Appendix

13.1 Formula Reference

Discount factor:

$$DF(t) = e^{-r(t)\cdot t}$$

Forward rate:

$$FR(t_1, t_2) = \frac{DF(t_1) - DF(t_2)}{DF(t_2) \cdot (t_2 - t_1)}$$

Net cashflow:

$$CF_{\text{net},i} = N \cdot \frac{FR_i - r_{\text{fixed}}}{f}$$

Present value:

$$PV = \sum_{i=1}^{n} CF_{\text{net},i} \cdot DF(t_i)$$

Sensitivity:

$$\frac{\partial PV}{\partial DF_i} = CF_{\text{net},i}$$

Returns:

$$\text{Return}(t, i) = \ln\left[\frac{DF(t, t_i)}{DF(t-1, t_i)}\right]$$

Covariance:

$$\Sigma_{ij} = \frac{1}{m-1} \sum_{t=2}^{m+1} [\text{Return}(t, i) - \mu_i] \cdot [\text{Return}(t, j) - \mu_j]$$

Portfolio variance:

$$\sigma_P^2 = \mathbf{w}^T \Sigma \mathbf{w}$$

Value at Risk:

$$VaR_\alpha = z_\alpha \cdot \sigma_P \cdot \sqrt{h}$$

13.2 References

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