Value-at-Risk (VaR)

Theoretical Foundations and Calculation Methods

Risk Management in Finance

English

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1 Introduction to Value-at-Risk

1.1 Definition

Value-at-Risk (VaR) is a statistical risk measure that states the maximum loss of a portfolio or individual position within a defined time period at a given probability (confidence level).

Formally: the VaR at confidence level α is the threshold value that will not be exceeded as a loss with probability $(1 - \alpha)$.

$$P(\text{Loss} > \text{VaR}) = 1 - \alpha \tag{1}$$

Example

A VaR of EUR 1 million at a 99% confidence level and a holding period of 10 days means: with 99% probability, the loss over the next 10 days will not exceed EUR 1 million.

1.2 Historical Development

The VaR concept was developed at J.P. Morgan in the 1990s and popularised through the RiskMetrics system. Since the Basel Accord (Basel II/III), VaR has been a regulatory standard for banks to determine the required capital for market risks.

1.3 Key Parameters

Three parameters define every VaR:

Confidence level (α)
: Typically 95% or 99%. A higher confidence level leads to a higher VaR.

Holding period (T)
: The time horizon of the risk assessment. For trading portfolios often 1–10 days; longer for strategic positions.

Position value
: Market value of the position or portfolio under consideration.

2 Variance-Covariance Method (Parametric Approach)

2.1 Core Principle

The variance-covariance method (also called the delta-normal method) is based on the assumption that returns are normally distributed. VaR is calculated analytically from the volatility and the corresponding quantile of the standard normal distribution.

2.2 Mathematical Derivation

2.2.1 Step 1: Return Distribution

Assumption: the logarithmic returns r follow a normal distribution:

$$r \sim N(\mu, \sigma^2) \tag{2}$$

where μ is the expected value (often assumed to be 0 for short time horizons) and σ is the standard deviation (volatility).

2.2.2 Step 2: Quantile of the Normal Distribution

The α-quantile of the standard normal distribution z_α determines the critical value:

$$z_{0.95} = 1.645 \qquad z_{0.99} = 2.326 \tag{3}$$

2.2.3 Step 3: VaR Formula

For a position with value V and daily volatility σ:

$$\boxed{\text{VaR} = V \times \sigma \times z_\alpha \times \sqrt{T}} \tag{4}$$

The factor $\sqrt{T}$ (square root of the holding period) scales the daily volatility to the desired time period (square-root-of-time rule).

2.3 Portfolio Perspective

For a portfolio of n positions, the variance-covariance method accounts for correlations between positions:

$$\sigma^2_{\text{Portfolio}} = \sum_i \sum_j w_i w_j \sigma_i \sigma_j \rho_{ij} \tag{5}$$

In matrix notation:

$$\sigma^2_{\text{Portfolio}} = \mathbf{w}' \mathbf{\Sigma} \mathbf{w} \tag{6}$$

where $\mathbf{w}$ is the vector of weights, $\mathbf{\Sigma}$ is the covariance matrix, and ρ_{ij} is the correlation between position i and position j.

2.4 Advantages and Disadvantages

Advantages:

- Simple and fast computation
- Analytically closed-form solution
- Accounts for correlations in portfolios
- Well suited to linear risk factors

Disadvantages:

- Normality assumption often unrealistic (fat tails)
- Underestimates extreme losses
- Not suitable for non-linear instruments (options)
- Volatility and correlations can be time-varying

3 Historical Simulation

3.1 Core Principle

Historical simulation is a non-parametric approach that makes no assumptions about the distribution of returns. Instead, the empirical distribution of historical returns is used to estimate potential future losses.

3.2 Calculation Steps

1. **Data collection:** Gather historical returns for a defined period (e.g. 250 trading days).
2. **Portfolio revaluation:** Calculate the hypothetical portfolio value for each historical return.
3. **Profit/loss distribution:** Construct a distribution of hypothetical gains and losses.
4. **Sorting:** Sort the results from the largest loss to the largest gain.
5. **Quantile determination:** The VaR corresponds to the $(1-\alpha)$-quantile of the sorted losses.

3.3 Mathematical Formulation

Consider n historical observation days. For each position j (value V_j in the home currency, $V = \sum_j V_j$), let r_{ij} be the daily logarithmic return of the corresponding risk factor (e.g. exchange rate) on day i.

Step 1 – Scaling to the holding period. The daily return is scaled to T days with the square-root-of-time rule and the portfolio is revalued:

$$V_i = \sum_j V_j \cdot \exp\left(r_{ij}\sqrt{T}\right), \qquad i = 1, \ldots, n \tag{7}$$

Step 2 – Losses. The hypothetical loss (gains as negative losses) and the return over the holding period are:

$$L_i = V - V_i, \qquad R_i^{(T)} = \frac{V_i - V}{V} = -\frac{L_i}{V} \tag{8}$$

Step 3 – Sorting. The losses are sorted in descending order: $L_{(1)} \geq L_{(2)} \geq \ldots \geq L_{(n)}$. $L_{(1)}$ is the worst scenario (rank 1).

Step 4 – VaR. With

$$k = \lfloor n \cdot (1-\alpha) \rfloor \tag{9}$$

the quantile boundary lies between rank k and rank $k+1$ (for $n = 250$, $\alpha = 99\%$: $n(1-\alpha) = 2.5$). The historical VaR is the mean of these two ranks:

$$\boxed{\mathrm{VaR}_\alpha^{\mathrm{hist}} = \frac{L_{(k)} + L_{(k+1)}}{2}} \qquad (k \geq 1) \tag{10}$$

If $k = 0$ (too few observations for the confidence level), the worst scenario is used: VaR $= L_{(1)}$.

Step 5 – Expected Shortfall. The ES is the average of the k worst losses, i.e. of all scenarios beyond the quantile boundary:

$$\boxed{\mathrm{ES}_\alpha^{\mathrm{hist}} = \frac{1}{k}\sum_{i=1}^{k} L_{(i)}} \qquad (k \geq 1;\ \text{otherwise ES} = L_{(1)}) \tag{11}$$

Numerical example from the FX VaR calculator

Portfolio: USD 100,000, GBP 50,000 and CNY 200,000; home currency EUR, portfolio value V = EUR 176,049; sample data with $n = 250$ days; $\alpha = 99\%$; $T = 10$ days.

$k = \lfloor 250 \cdot 0.01 \rfloor = \lfloor 2.5 \rfloor = 2$

Rank	10-day return	Loss (EUR)	
1	−3.321 %	5,847	ES
2	−3.268 %	5,754	ES, VaR
3	−2.773 %	4,882	VaR

$$\mathrm{VaR}_{99\%,10d} = \frac{L_{(2)} + L_{(3)}}{2} = \frac{5{,}754 + 4{,}882}{2} = \textbf{EUR 5,318}$$

$$\mathrm{ES}_{99\%,10d} = \frac{L_{(1)} + L_{(2)}}{2} = \frac{5{,}847 + 5{,}754}{2} \approx \textbf{EUR 5,800}^{1}$$

Check: 2.773 %× EUR 176,049 ≈ EUR 4,882.

3.4 Advantages and Disadvantages

Advantages:

- No distributional assumptions required
- Automatically captures fat tails and skewness
- Implicit treatment of correlations
- Intuitive and easy to communicate

Disadvantages:

- Dependent on the quality and length of historical data
- Assumes the past is representative of the future
- Cannot anticipate structural market changes
- Ghost effects: extreme events drop out of the calculation after T days

[1] Differences in the last digit are due to rounding of the individual losses.

4 Monte Carlo Simulation

4.1 Core Principle

Monte Carlo simulation generates a large number of random scenarios based on a stochastic model. It is the most flexible VaR method and can value complex portfolios containing non-linear instruments.

4.2 Calculation Steps

1. **Model specification:** Choose a stochastic model for the risk factors (e.g. geometric Brownian motion).
2. **Parameter estimation:** Estimate parameters from historical data (drift, volatility, correlations).
3. **Random number generation:** Generate correlated random numbers (e.g. using Cholesky decomposition).
4. **Scenario generation:** Simulate n price paths (typically 10,000–100,000).
5. **Portfolio valuation:** Value the portfolio for each scenario.
6. **VaR determination:** Determine the $(1-\alpha)$-quantile of the simulated losses.

4.3 Stochastic Model

The most common model is geometric Brownian motion (GBM):

$$dS = \mu S\,dt + \sigma S\,dW \tag{12}$$

In discrete form for a time step Δt:

$$S(t+\Delta t) = S(t) \times \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t} \times \varepsilon\right] \tag{13}$$

where $\varepsilon \sim N(0,1)$ is a standard normally distributed random variable.

4.4 Correlated Risk Factors

For multiple correlated risk factors, Cholesky decomposition is used:

$$\mathbf{\Sigma} = \mathbf{L} \times \mathbf{L}' \tag{14}$$

where $\mathbf{L}$ is a lower triangular matrix. Correlated random numbers are generated by:

$$\mathbf{z}_{\text{correlated}} = \mathbf{L} \times \mathbf{z}_{\text{independent}} \tag{15}$$

4.5 Advantages and Disadvantages

Advantages:

- Greatest flexibility in model choice
- Suitable for non-linear instruments (options, derivatives)
- Can model complex dependence structures
- Enables scenario analysis and stress testing

Disadvantages:

- High computational cost
- Model risk: results depend strongly on the chosen model
- Sampling error with an insufficient number of simulations
- Complex implementation and validation

5 Expected Shortfall (CVaR)

5.1 Definition and Motivation

The Expected Shortfall (ES), also known as Conditional Value-at-Risk (CVaR), is a coherent risk measure that addresses the weaknesses of VaR. It gives the expected loss given that the loss exceeds the VaR.

Mathematically:

$$\mathrm{ES}_\alpha = E[\mathrm{Loss} \mid \mathrm{Loss} > \mathrm{VaR}_\alpha] \tag{16}$$

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha} \times \int_\alpha^1 \mathrm{VaR}_u \, du \tag{17}$$

5.2 Calculation

Under normality:

$$\mathrm{ES}_\alpha = \sigma \times \frac{\varphi(z_\alpha)}{1-\alpha} \tag{18}$$

where $\varphi(z)$ is the density function of the standard normal distribution.

Under historical simulation, the ES is the average of the $k = \lfloor n(1-\alpha) \rfloor$ worst losses (definition and numerical example in Section 3.3):

$$\mathrm{ES} = \frac{1}{k} \times \sum_{i=1}^{k} L_{(i)} \tag{19}$$

In the example ($n = 250$, $\alpha = 99\%$, $k = 2$) this gives ES $= (5{,}847 + 5{,}754)/2 \approx$ EUR 5,800, compared with a VaR of EUR 5,318.

5.3 Advantages over VaR

- **Coherent risk measure:** Satisfies sub-additivity ($\mathrm{ES}(A+B) \leq \mathrm{ES}(A)+\mathrm{ES}(B)$).
- **Tail risk:** Accounts for the severity of extreme losses, not merely their frequency.
- **Regulatory:** Basel III increasingly replaces VaR with ES for market risk.
- **Convexity:** Enables efficient portfolio optimisation.

6 Method Comparison

The following table summarises the key properties of the three VaR methods:

Criterion	Variance-Covariance	Hist. Simulation	Monte Carlo
Distribution assumption	Normal	None (empirical)	Selectable
Computational cost	Low	Medium	High
Fat tails	Not captured	Captured	Model-dependent
Non-linear instruments	Not suitable	Conditionally suitable	Well suited
Implementation	Simple	Simple	Complex
Model risk	Medium	Low	High

Table 1: Comparison of the three VaR methods

7 Limitations of VaR

Despite its widespread use, VaR has fundamental weaknesses.

7.1 Lack of Sub-additivity

VaR is not a coherent risk measure because it is not sub-additive. In rare cases, $\text{VaR}(A+B) > \text{VaR}(A) + \text{VaR}(B)$, which would appear to penalise diversification.

7.2 Tail Blindness

VaR says nothing about the severity of losses beyond the threshold. Two portfolios with the same VaR can have completely different tail risks.

7.3 Pro-cyclicality

In times of crisis, VaR rises, which can lead to selling pressure that exacerbates the crisis. This pro-cyclical behaviour can amplify systemic risks.

8 Application Example: FX Portfolio

Consider an example of VaR calculation for a foreign exchange position.

8.1 Initial Situation

- Position: USD 1,000,000 against UZS
- Current rate: 12,500 UZS/USD
- Daily volatility: $\sigma = 0.3\%$ (based on historical data)
- Confidence level: 99%
- Holding period: 10 days

8.2 Calculation (Variance-Covariance)

Position value in UZS:

$$V = 1{,}000{,}000 \times 12{,}500 = 12{,}500{,}000{,}000 \text{ UZS} \tag{20}$$

VaR calculation:

$$\begin{aligned} \text{VaR}_{99\%,10d} &= V \times \sigma \times z_{0.99} \times \sqrt{10} && (21)\\ &= 12{,}500{,}000{,}000 \times 0.003 \times 2.326 \times 3.162 && (22)\\ &\approx 276{,}000{,}000 \text{ UZS} && (23) \end{aligned}$$

8.3 Interpretation

With 99% probability, the loss from this USD position will not exceed 276 million UZS over the next 10 trading days. In other words, only in 1% of cases (roughly 2.5 days per year) should a higher loss be expected.

9 Further Reading

- Jorion, P. (2007): *Value at Risk: The New Benchmark for Managing Financial Risk*. 3rd ed., McGraw-Hill.
- Hull, J. C. (2018): *Risk Management and Financial Institutions*. 5th ed., Wiley.
- McNeil, A. J., Frey, R., Embrechts, P. (2015): *Quantitative Risk Management*. Princeton University Press.
- Basel Committee on Banking Supervision (2019): *Minimum Capital Requirements for Market Risk*. BIS.
- Artzner, P. et al. (1999): Coherent Measures of Risk. *Mathematical Finance* 9(3), 203–228.

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